Victorian Summer School in Ultracold Physics

31st January 2014
Melbourne, Victoria, Australia

Topological excitations and 2D quantum turbulence

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OUTLINE

- some classical fluid dynamics
  - Navier-Stokes equation
  - vortices and turbulence
  - cascades and energy spectra

- superfluids and quantized vortices
  - Gross-Pitaevskii hydrodynamics
  - quantum/superfluid vortices
  - point vortex approximation

- topics in superfluid 2D turbulence
  - statistical hydrodynamics and Onsager vortices
  - negative absolute temperatures
  - evaporative heating mechanism

some reference texts

A.J. Leggett,
“Quantum Liquids”,
(Oxford University Press 2006).

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(Cambridge University Press 2001).

P. G. Saffman,
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C.F. Barenghi, R.J. Donnelly, W.F. Vinen eds,
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Rep. Prog. Phys. **43**, 35(1980).

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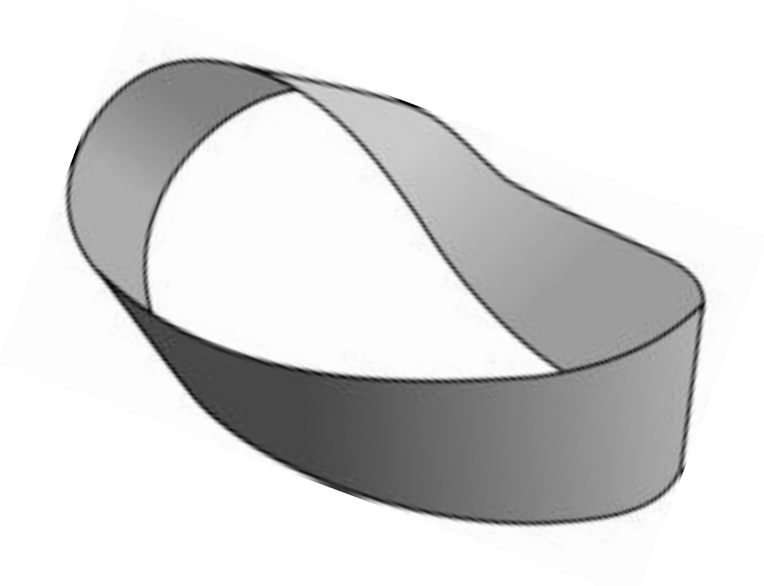
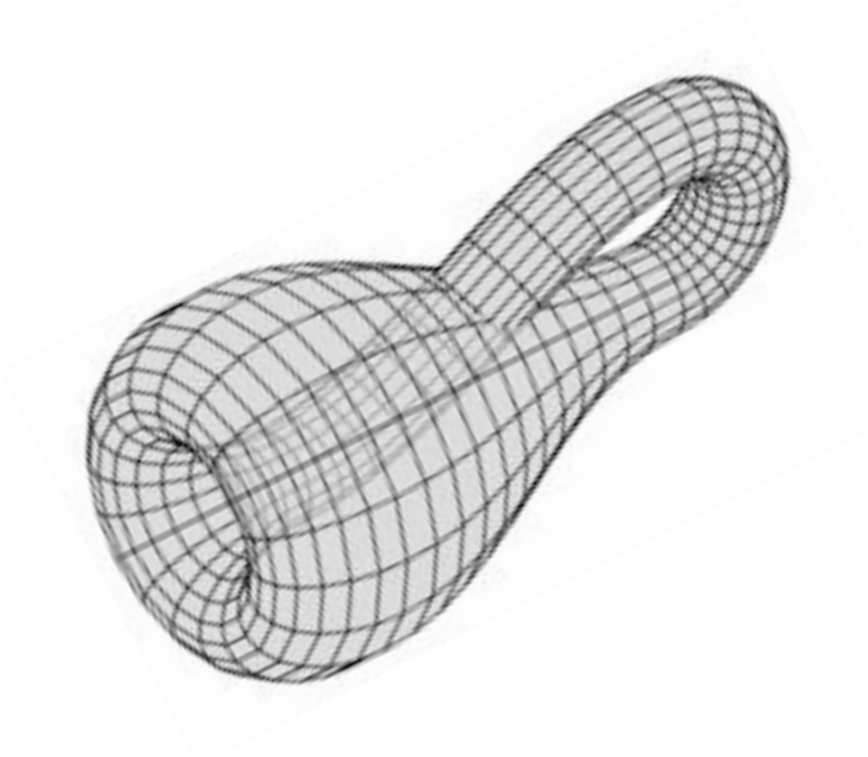
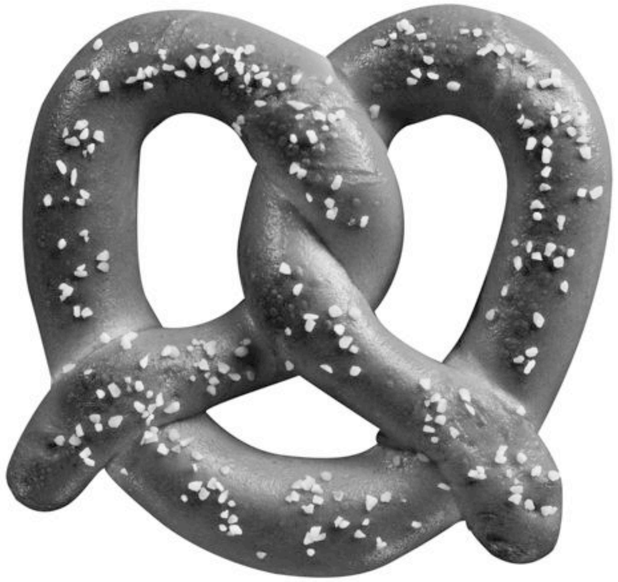
“Onsager and the theory of hydrodynamic turbulence”,
Rev. Mod. Phys. **78**, 87 (2006).

topology

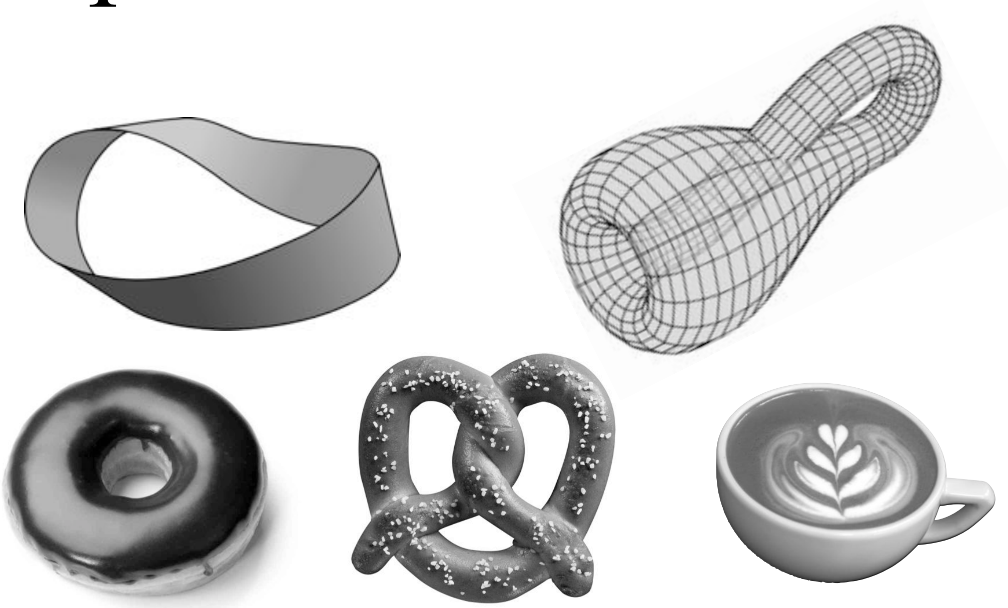






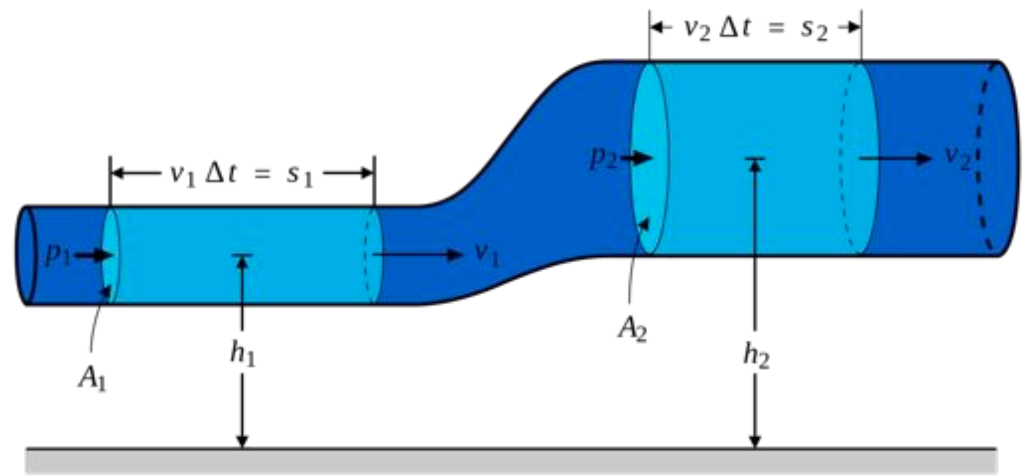
Clickerless clicker question: Which of these objects are topologically "equivalent" (homeomorphic)?

A pretzel and Klein bottle
B coffee cup, Möbius strip and doughnut
C pretzel and doughnut
D coffee cup and Klein bottle



basics of fluids

$$\sum F = ma$$



$$\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{u} + \mathbf{f}$$

incompressible Navier-Stokes

$$\nabla \cdot \mathbf{u} = 0$$

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}$$

vorticity

invicid if kinematic viscocity $\nu = 0$

Helmholz decomposition

$$\mathbf{u} = \mathbf{u}_{\|} + \mathbf{u}_{\perp} \qquad \nabla \cdot \mathbf{u}_{\perp} = 0 \qquad \nabla \times \mathbf{u}_{\|} = 0$$

$$\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{\nabla p}{\rho} + \cancel{\nu\nabla^2\mathbf{u}} + \mathbf{f}$$

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla u^2/2 - \mathbf{u}\times(\nabla\times\mathbf{u})$$

vector identity

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u}\times(\nabla\times\mathbf{u}) = -\nabla\left(\frac{1}{2}u^2 + \frac{p}{\rho}\right)$$

$$\boldsymbol{\omega} = \nabla\times\mathbf{u}$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla\times(\mathbf{u}\times\boldsymbol{\omega}) = -\mathbf{u}\cdot\nabla\boldsymbol{\omega} + \boldsymbol{\omega}\cdot\nabla\mathbf{u}$$

vorticity equation

conservation laws

$$W = \int \boldsymbol{\omega}\, d^3\mathbf{r}$$ total vorticity $$\frac{dW}{dt} = 0$$

$$\Omega = \int \omega^2\, d^2\mathbf{r}$$ total enstrophy $$\frac{d\Omega}{dt} = 0$$ only in 2D!

$$P = \rho \int \mathbf{u}\, d^3\mathbf{r}$$ total momentum $$\frac{dP}{dt} = 0$$

$$E = \frac{1}{2}\rho \int u^2\, d^3\mathbf{r}$$ total energy $$\frac{dE}{dt} = 0$$

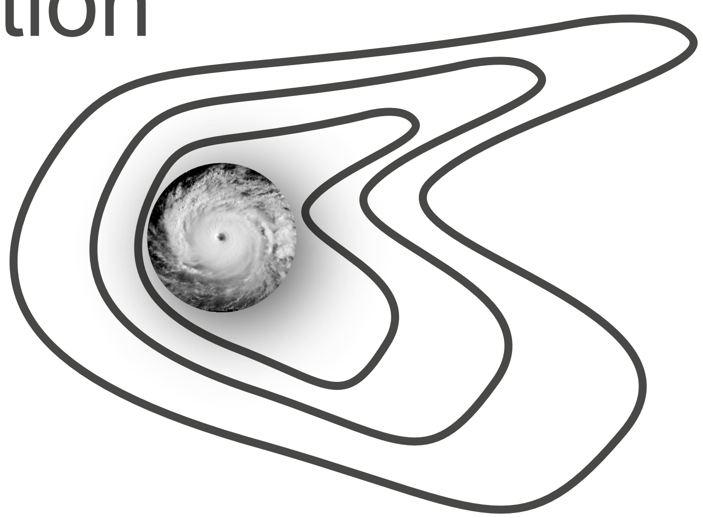
theorems of Kelvin and Helmholz

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}$$

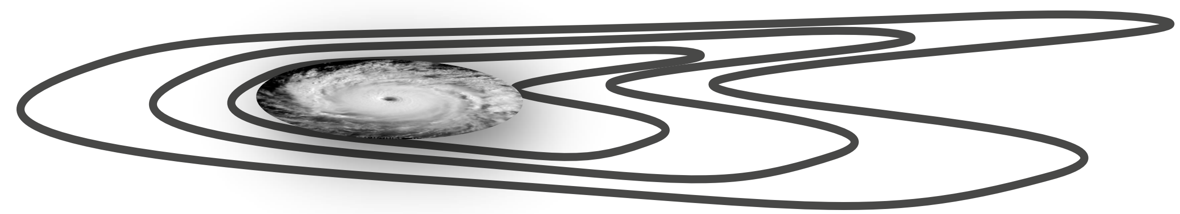
$$\Gamma = \oint \mathbf{u} \cdot d\mathbf{l} = \int \boldsymbol{\omega} \cdot d\mathbf{S}$$

$$\frac{d\Gamma}{dt} = 0$$

circulation



compare Liouville theorem



$$H = \int \mathbf{u} \cdot \boldsymbol{\omega} \, d^3\mathbf{r}$$

$$\frac{dH}{dt} = 0$$

helicity

a vortex line is everywhere tangential to vorticity vector

must form a closed loop or end at a boundary (infinity)

$$\nabla \cdot \boldsymbol{\omega} = 0$$

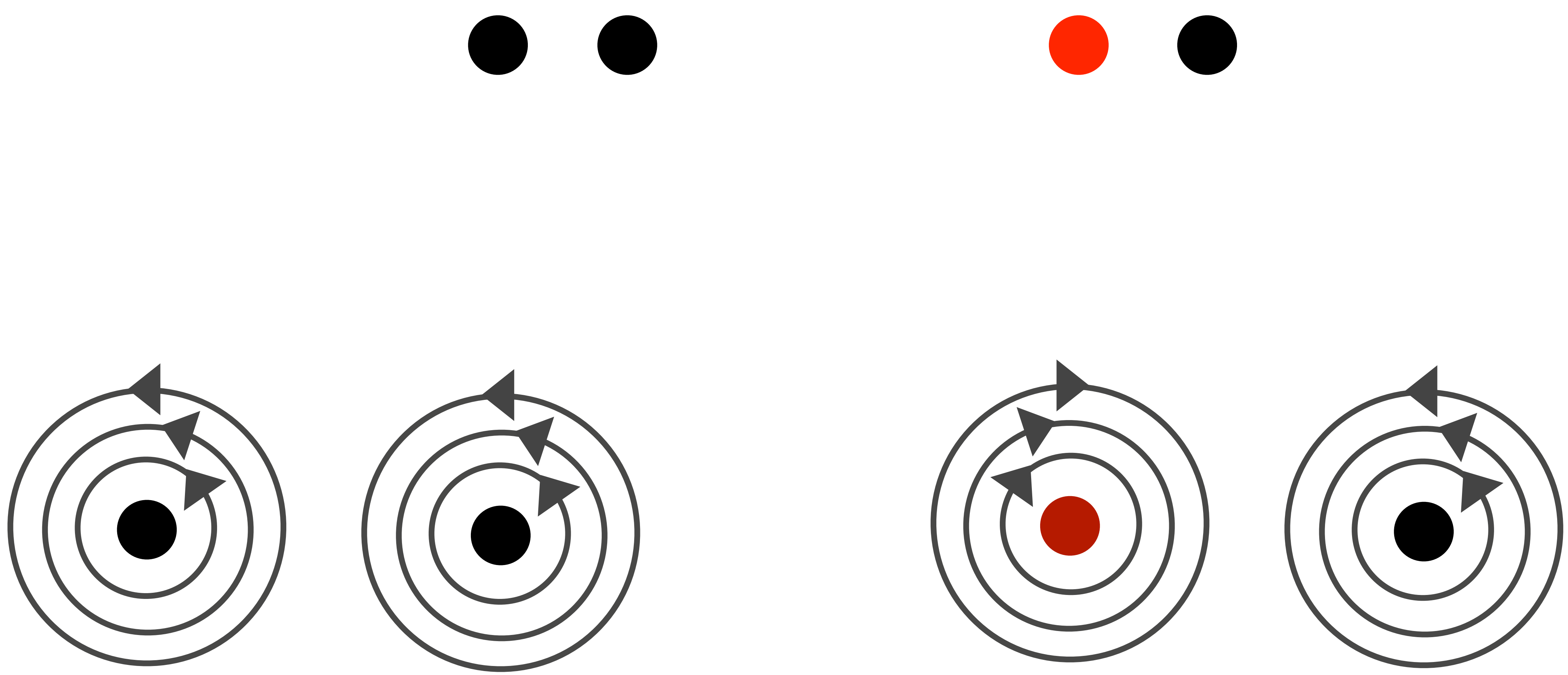
moves with the fluid it is embedded in

$$\frac{d\Gamma}{dt} = 0$$

some properties of vortices

vortex-antivortex pair (vortex ring)

translates at constant velocity v ~ 1 / d

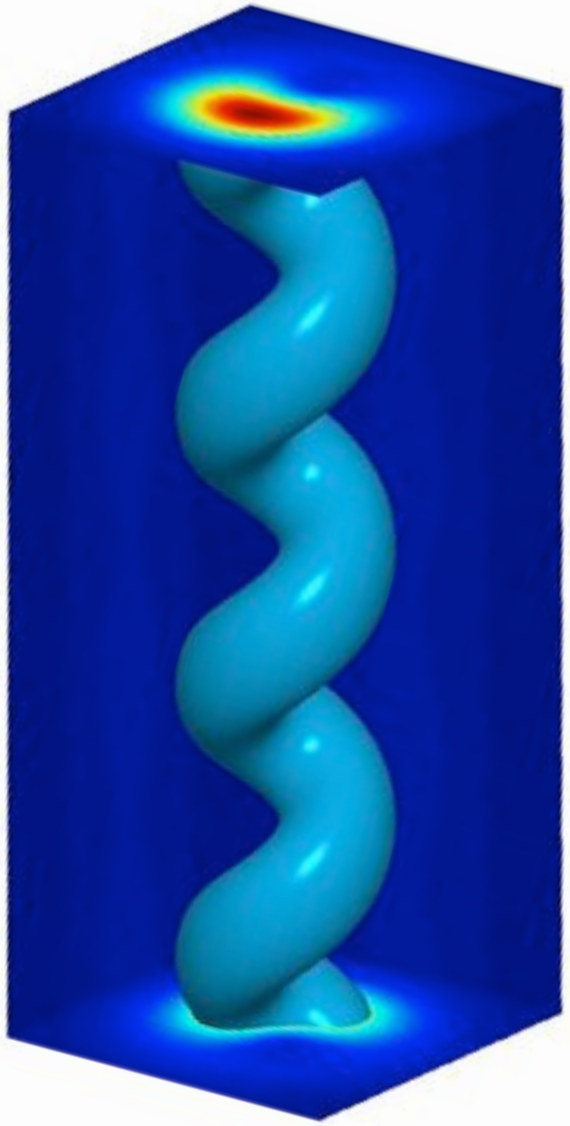


Kelvin waves: helical perturbations of a vortex line

Thomson (Lord Kelvin), Philos. Mag. 10, 155 (1880)

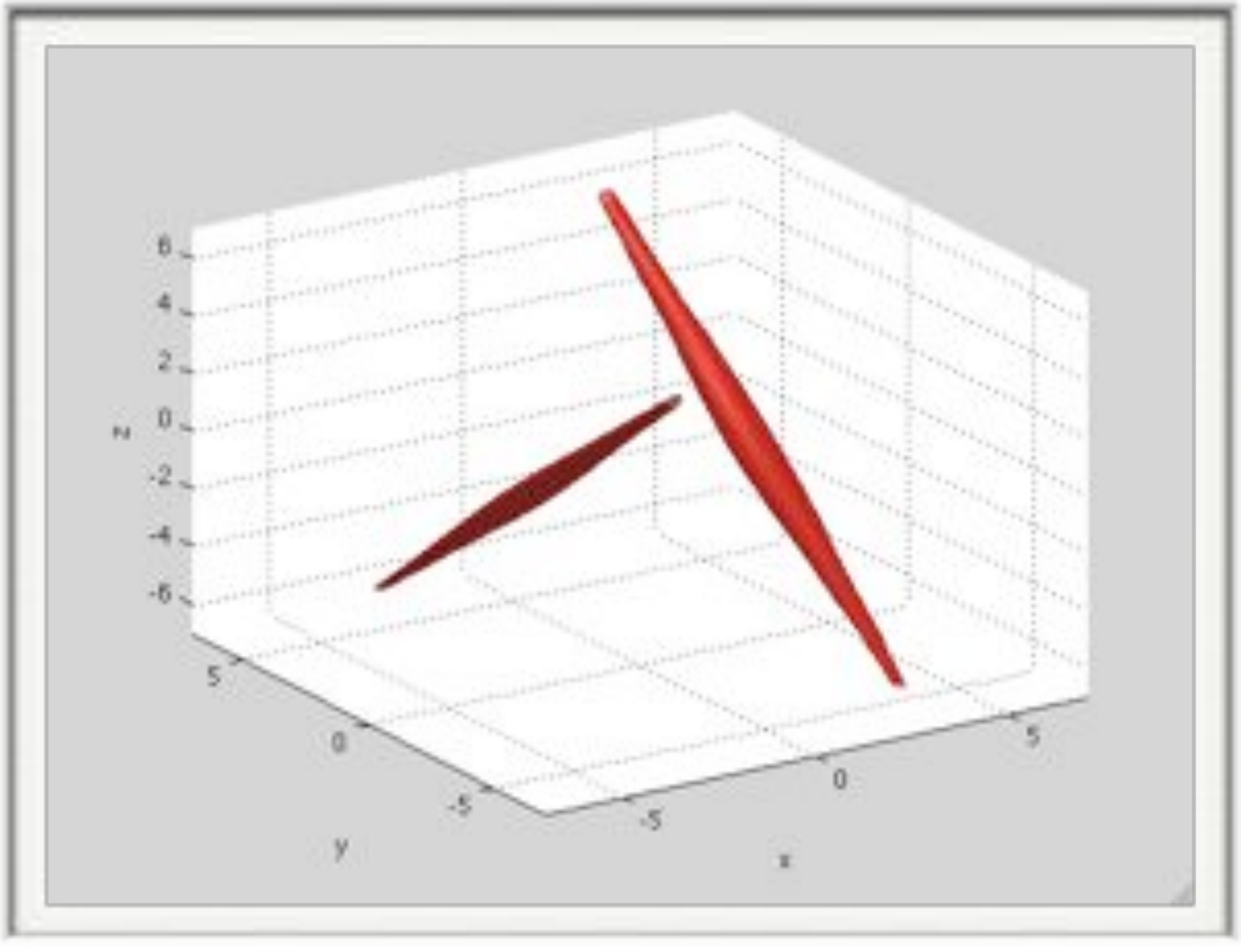
$$\omega^{\pm}(k) = \frac{\Gamma}{2\pi a^2}\left(1 \pm \sqrt{1 + ka\frac{K_0(ka)}{K_1(ka)}}\right)$$

$$\omega(k) \approx \frac{\Gamma k^2}{4\pi}\log\left(\frac{1}{ka}\right)$$

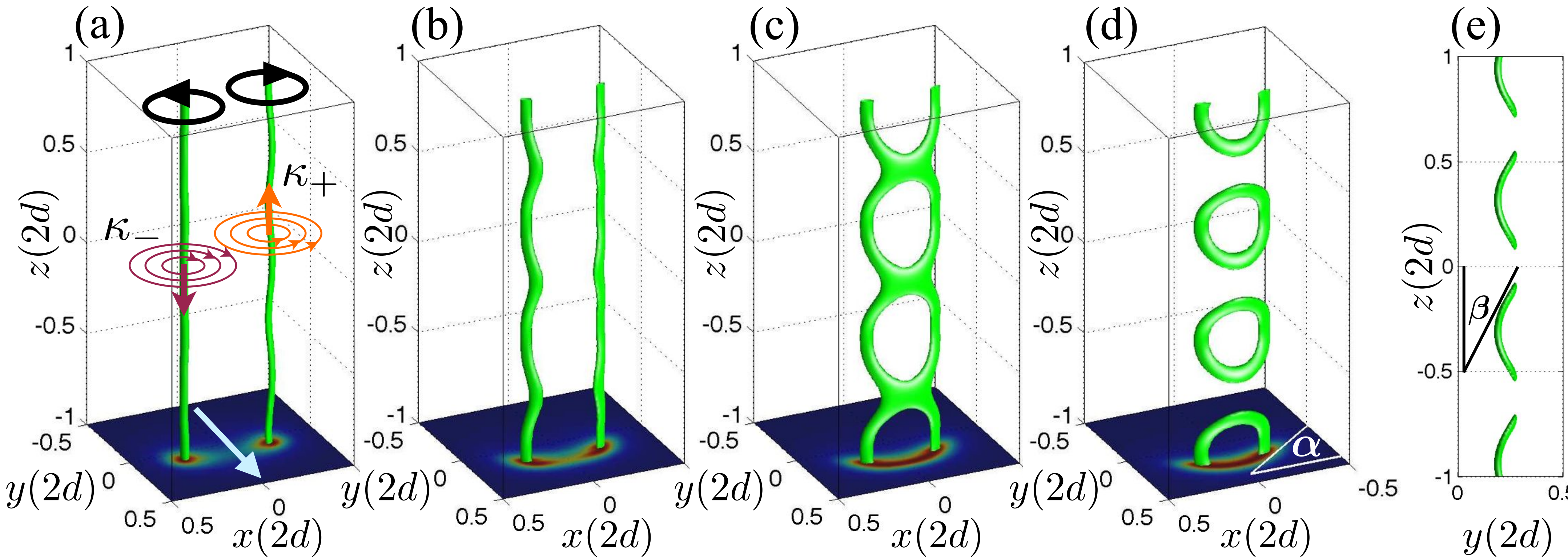


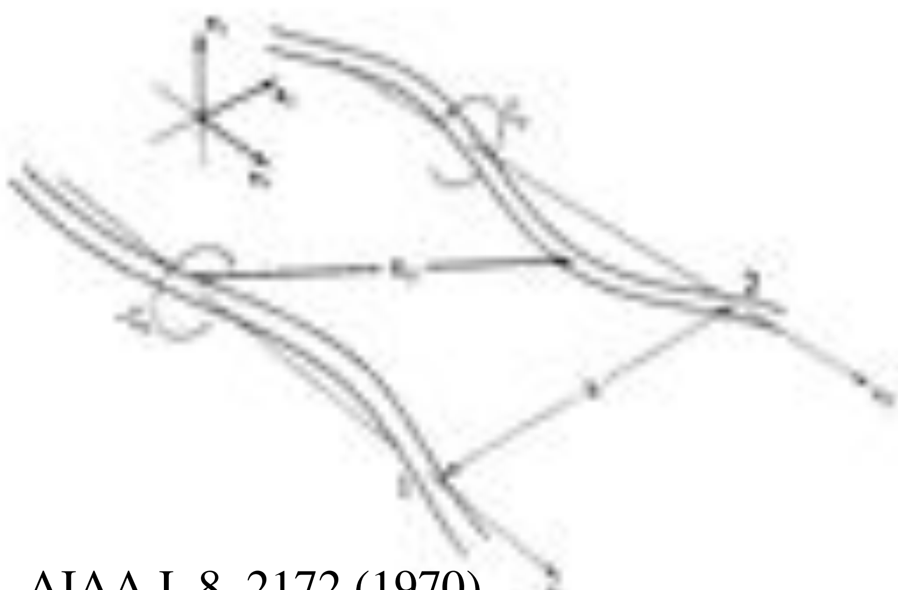
some statements regarding vortices

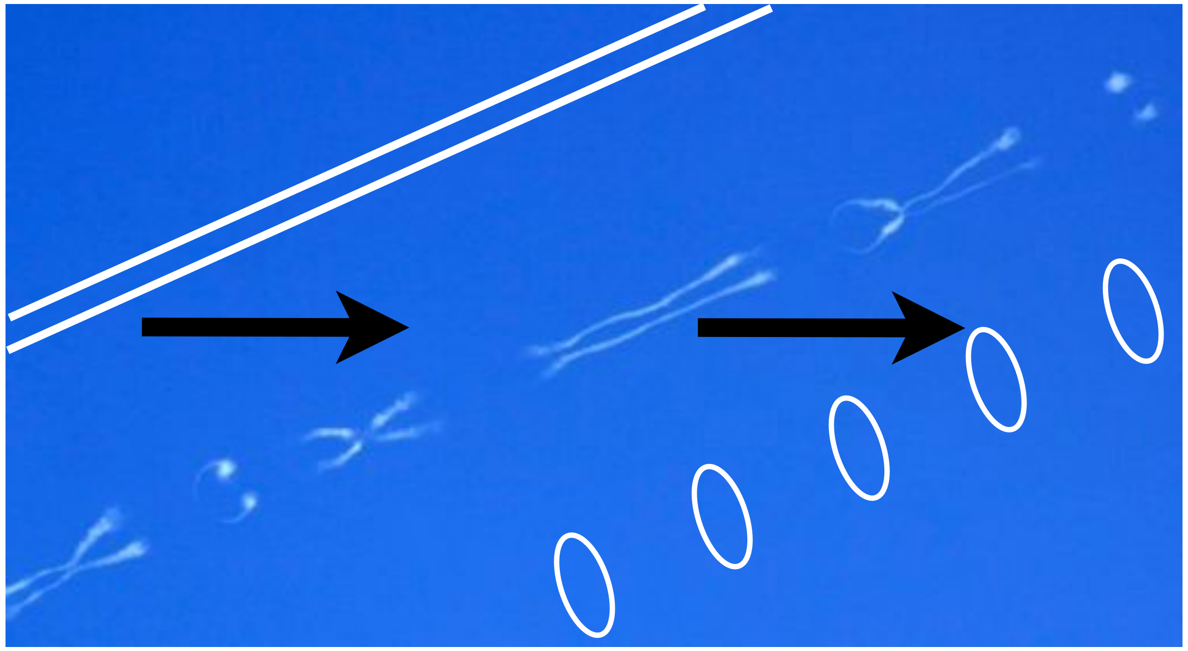
vortex reconnection / pair annihilation



Crow instability

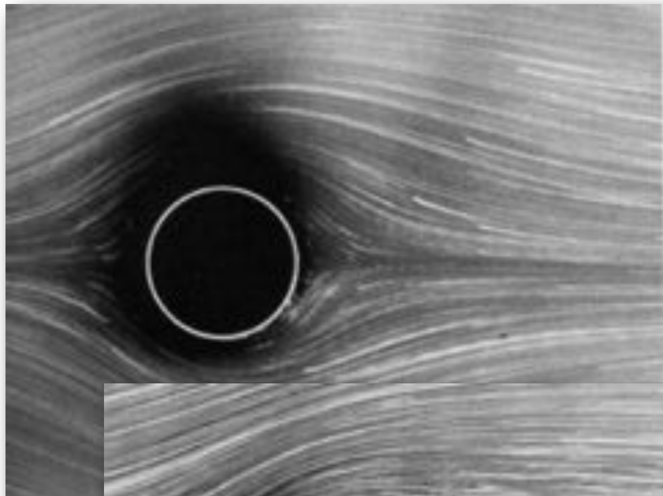
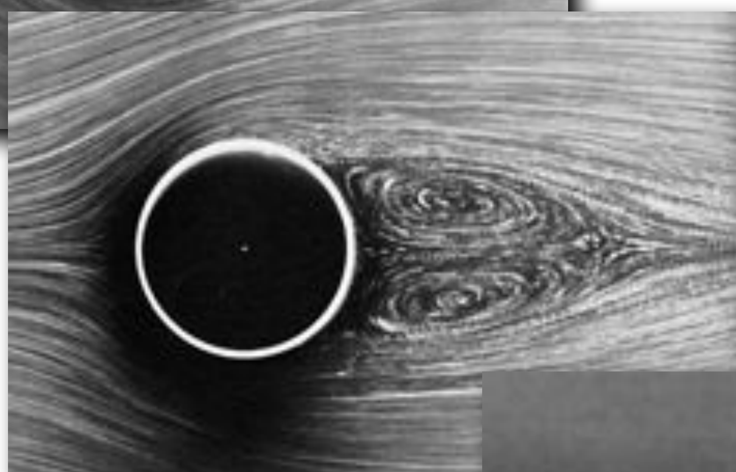






AIAA J. 8, 2172 (1970)
Phys. Rev. A 84, 021603(R) (2011)

phenomenology of turbulence: symmetry breakings

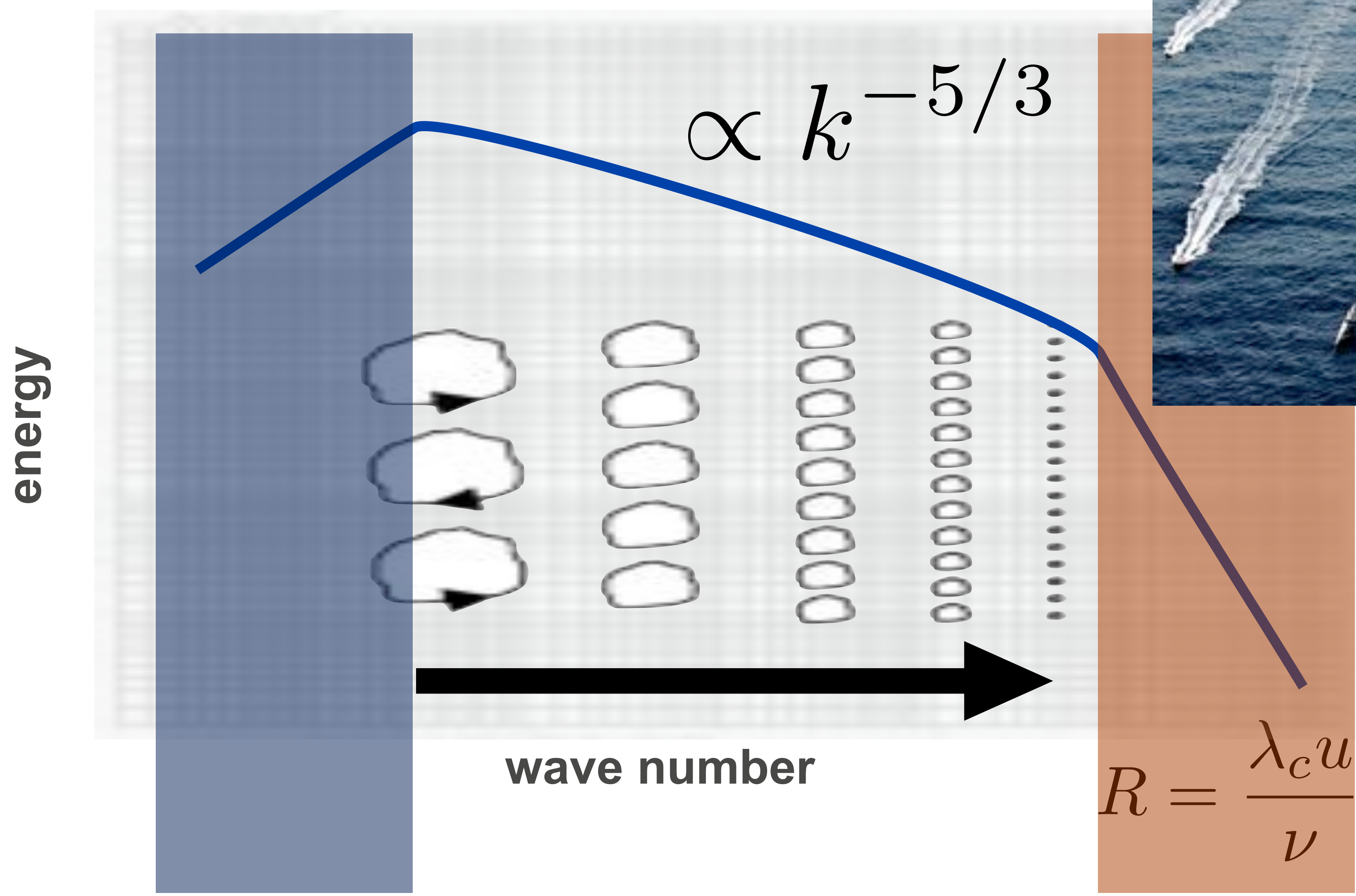
Reynolds number $R \propto \dfrac{(\mathbf{u}\cdot\nabla)\mathbf{u}}{\nu\nabla^2\mathbf{u}}$

$$R = \frac{Lu}{\nu}$$

expect turbulence for $R \gtrsim 1000$


phenomenology of 3D turbulence: energy spectrum

3D - direct energy cascade



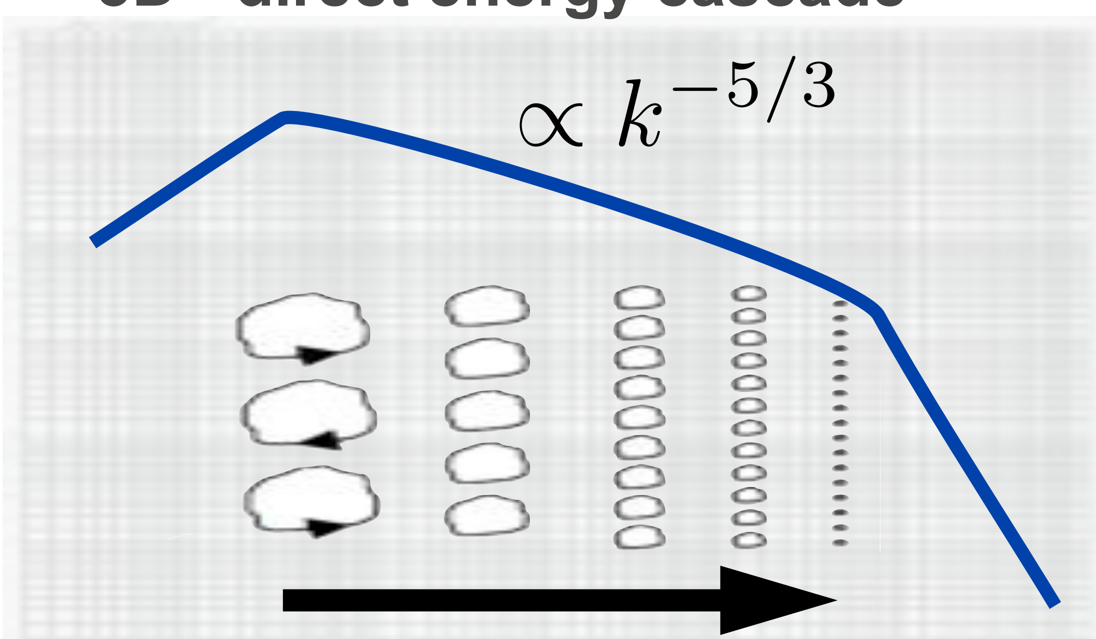
energy injection scale **inertial range** **dissipation scale**

phenomenology of 3D turbulence: energy spectrum

Richardson

Kolmogorov

3D - direct energy cascade



$$E(k) = C\epsilon^{\alpha}k^{\beta}$$

$$E = \int E(k)dk \qquad \epsilon = -\frac{\partial E}{\partial t}$$

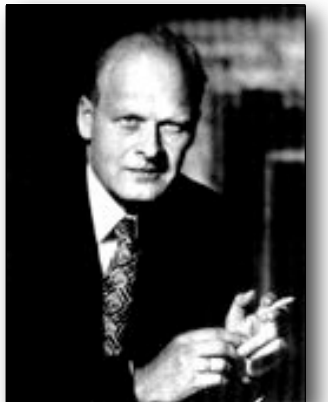
$$[E(k)] = \frac{L^3}{T^2} \qquad [\epsilon] = \frac{L^2}{T^3} \qquad [k] = \frac{1}{L}$$

$$3 = 2\alpha - \beta$$
$$2 = 3\alpha$$

$$\alpha = 2/3 \qquad \beta = -5/3$$

phenomenology of 2D turbulence: energy spectra

Onsager 1949 Kraichnan 1967



$$E_\Omega(k) \propto k^{-2}\Omega(k)$$

enstrophy conserved!

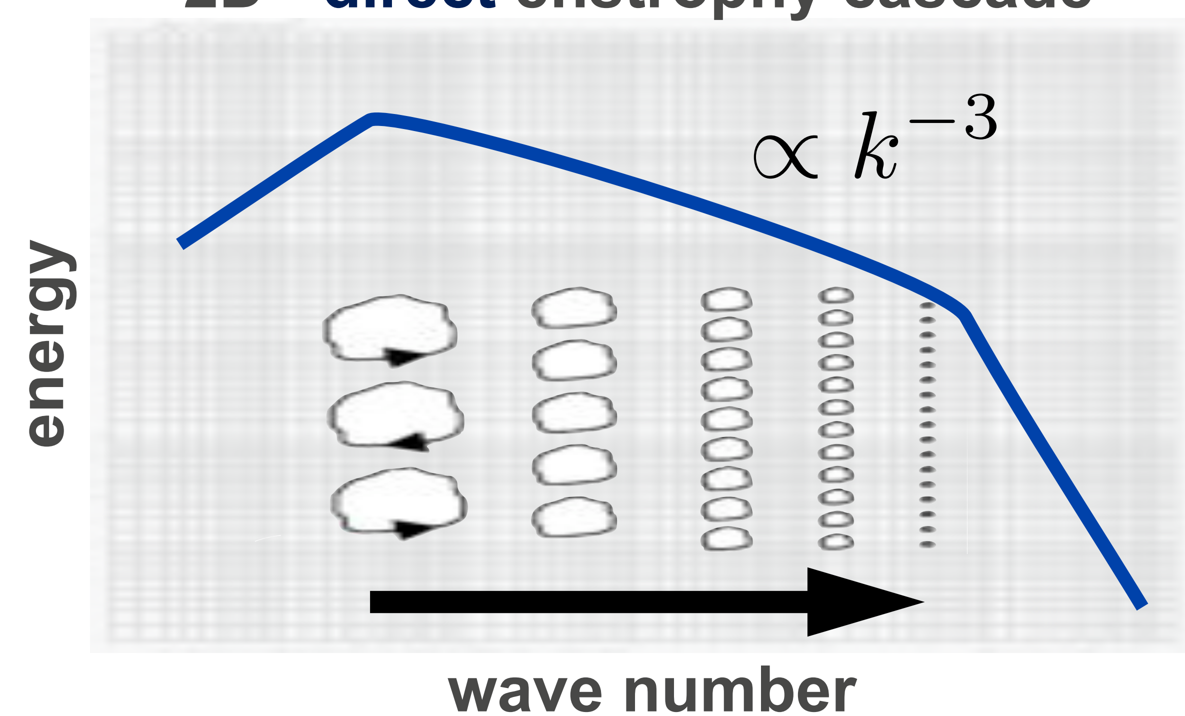
$$\Omega = \int \omega^2 \, d^2\mathbf{r} \qquad \boldsymbol{\omega} = \nabla \times \mathbf{u}$$

$$\Omega(k) = C'\eta^\alpha k^\beta \qquad \eta = -\frac{\partial\Omega}{\partial t}$$

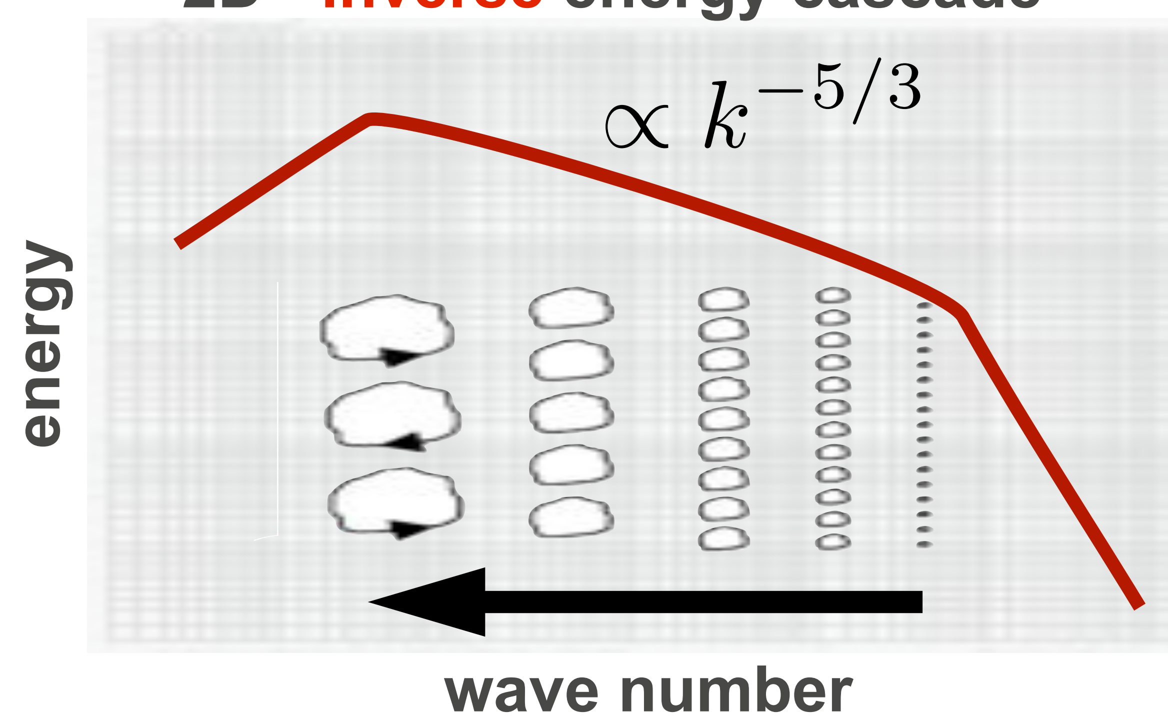
$$[\Omega(k)] = \frac{L}{T^2} \qquad [\eta] = \frac{1}{T^3}$$

$$\alpha = 2/3 \qquad \beta = -1$$

2D - **direct** enstrophy cascade



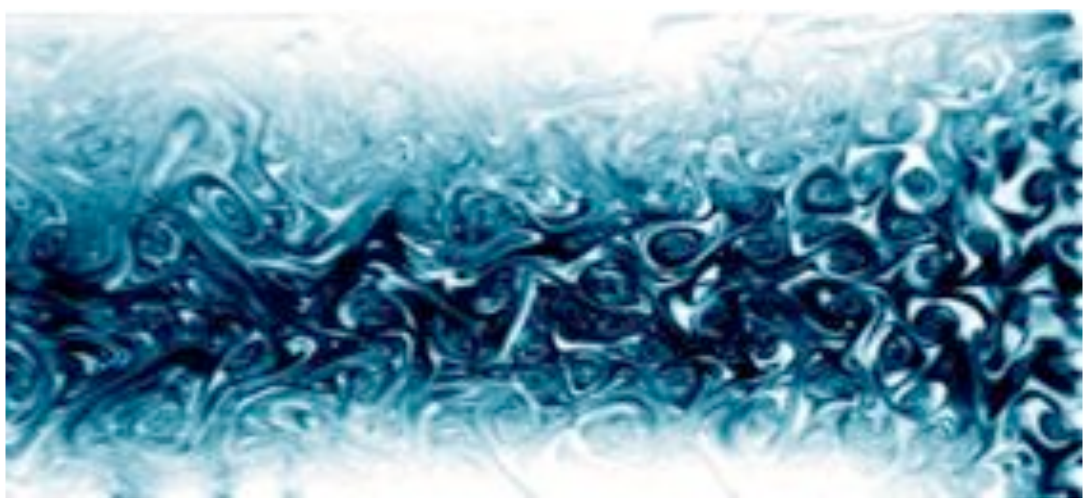
2D - **inverse** energy cascade



major prediction of 2D turbulence: "Onsager vortices"

Onsager 1949

- geostrophic flows
- soap films
- computer simulations
- Jovian atmosphere



potentially particularly relevant to superfluids...

Onsager 1949:

"Vortices in a superfluid are presumably quantized; the quantum of circulation is h/m, where m is the mass of a single molecule"

quantum/superfluid turbulence!

Clickerless clicker question:

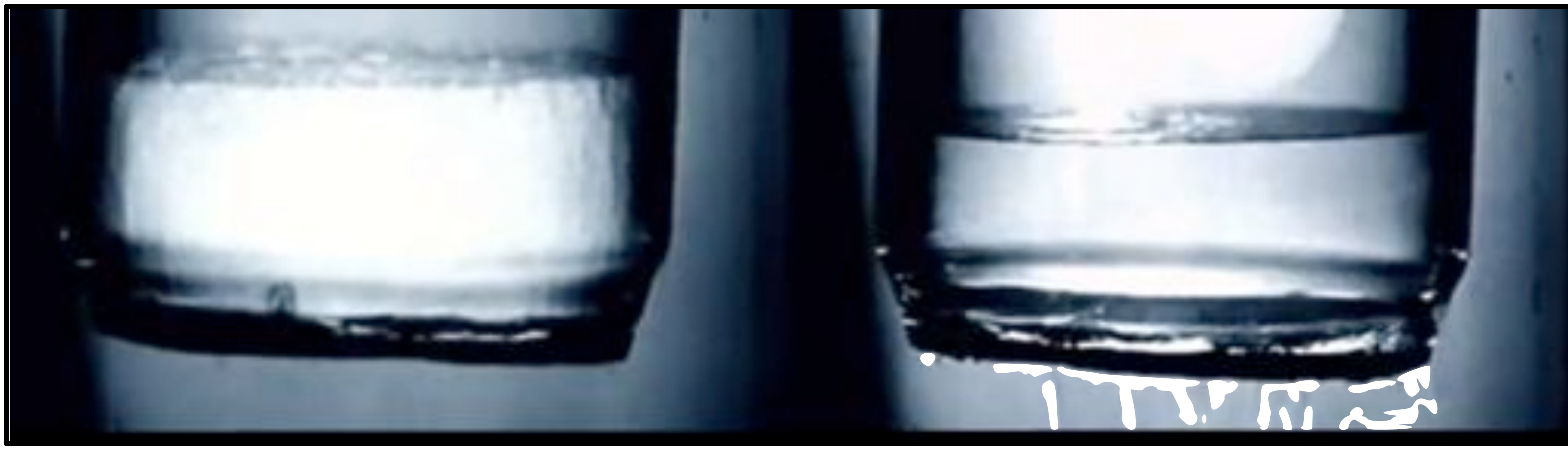
A need a quick break now

B keep going till the end

C "Please stop. I'm bored. Please stop. I'm bored."

superfluids are unlike normal fluids

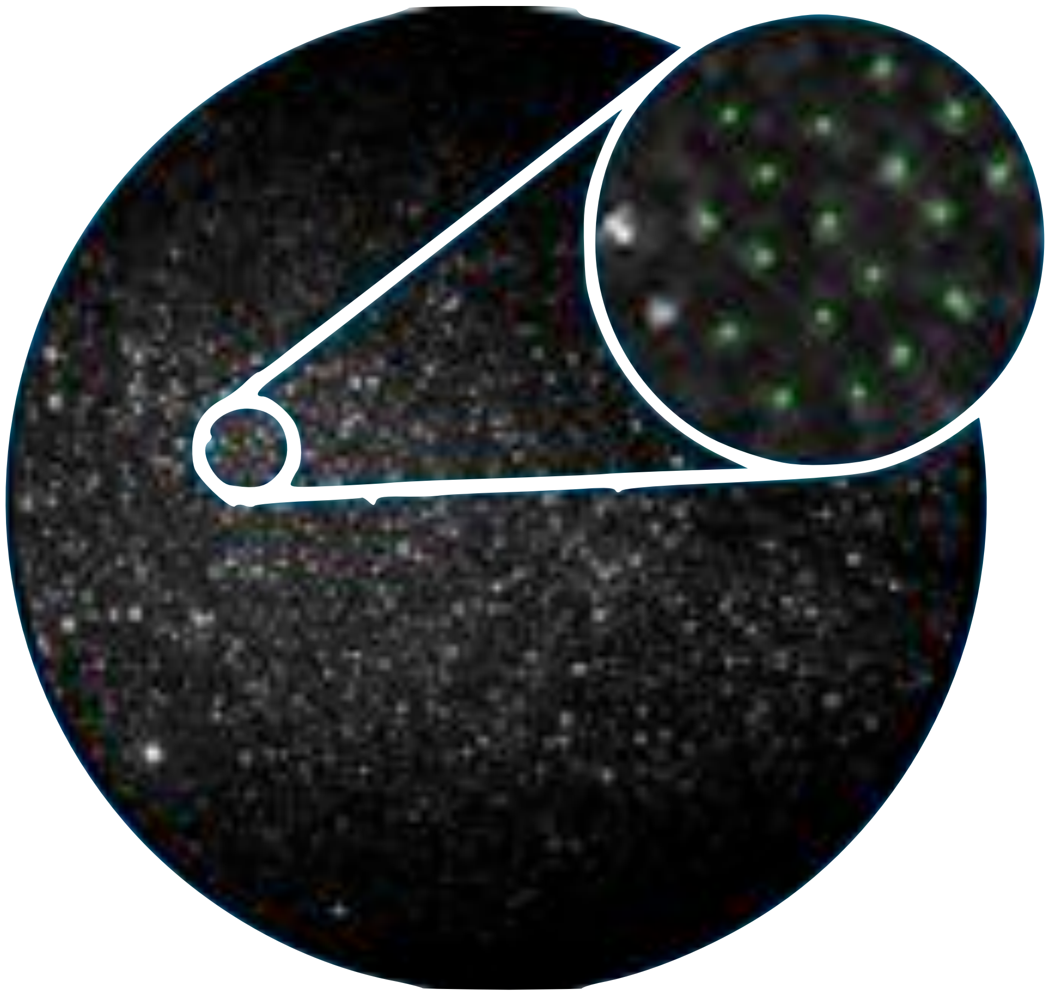
normal fluid | superfluid

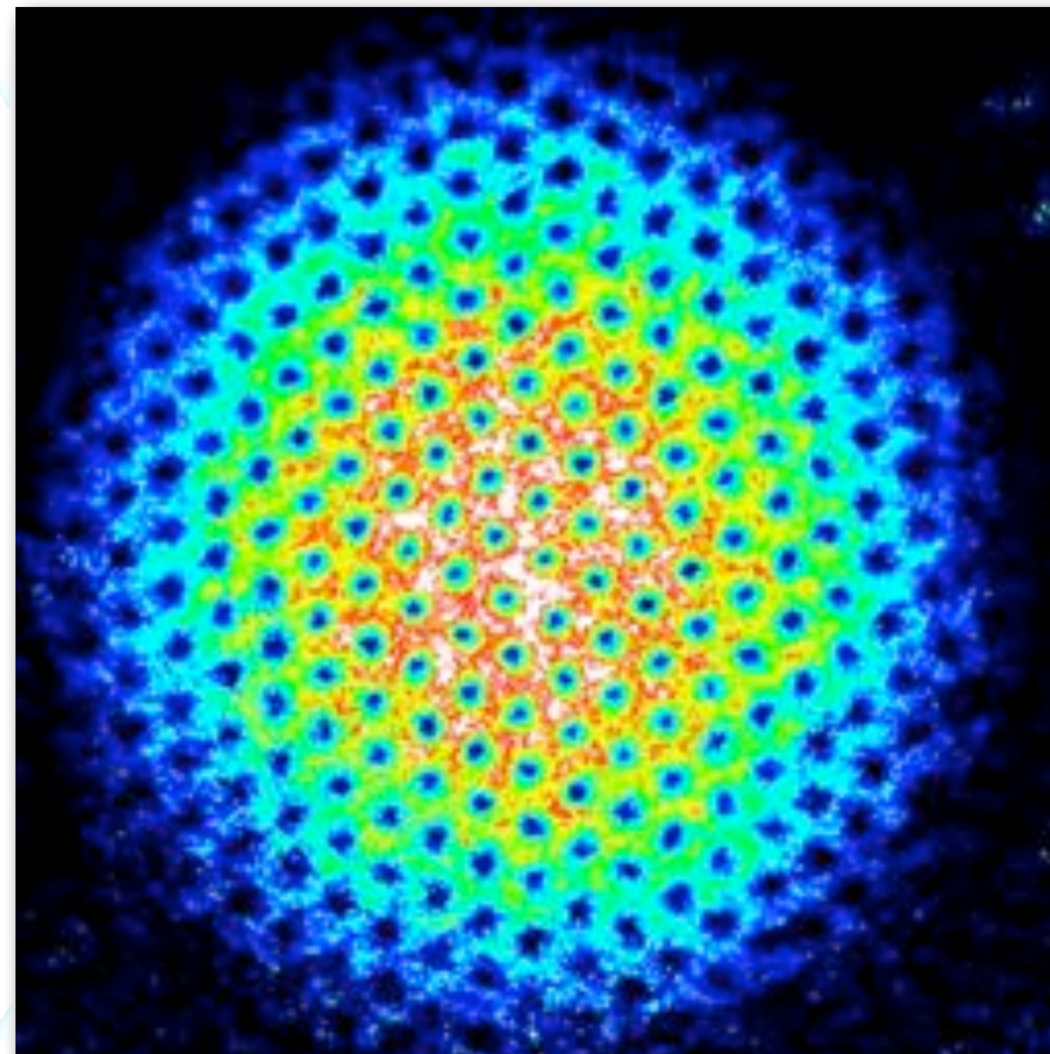


"frictionless flow"

quantized vortices

helium II | **atomic BEC**





core size of vortices

Ångstöm | **micron**

"Superfluidity is a complex of phenomena"

superfluids can be modelled by
"macroscopic wavefunctions"

Gross-Pitaevskii equation / nonlinear Schrödinger equation

$$i\hbar\frac{\partial\phi(\mathbf{r},t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\mathrm{ext}}(\mathbf{r}) + g|\phi(\mathbf{r},t)|^2\right)\phi(\mathbf{r},t)$$

a simple model that works well e.g. for atomic BECs

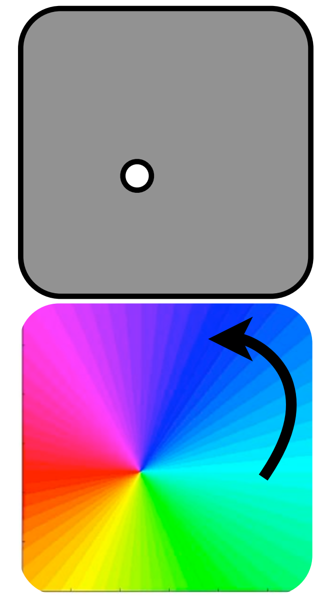
quantization of vorticity: “topological excitations”

postulate of quantum mechanics

$$P(\mathbf{r}) = \int \phi^*(\mathbf{r})\phi(\mathbf{r})dV$$

$$\phi(\mathbf{r}) = |\phi(\mathbf{r})|\, e^{i\theta(\mathbf{r})}$$

$$n(\mathbf{r}) = |\phi(\mathbf{r})|^2$$



$$v_s(\mathbf{r}) \equiv \frac{\hbar}{m}\nabla\theta(\mathbf{r})$$

$$\theta(\mathbf{r}) = \arctan(y/x)$$

$$v = \frac{h}{m}\frac{\ell}{r}$$

$$\oint \mathbf{v}_s \cdot d\mathbf{l} = n\kappa$$

compare Kelvin theorem:

quantization versus conservation of circulation

$$\kappa = \frac{h}{m}$$

quantum of circulation

$$n$$

integer winding number / vortex charge

geometric (Berry) phase

quantum hydrodynamics

$$i\hbar\frac{\partial\phi(\mathbf{r},t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\mathrm{ext}}(\mathbf{r}) + g|\phi(\mathbf{r},t)|^2\right)\phi(\mathbf{r},t)$$

Madelung transformation $\phi(\mathbf{r},t) = |\phi(\mathbf{r},t)|e^{i\theta(\mathbf{r},t)}$ $\quad n_c(\mathbf{r},t) = |\phi(\mathbf{r},t)|^2$

$$\mathbf{j}(\mathbf{r},t) = \frac{\hbar}{2im}\left(\phi^*(\mathbf{r},t)\nabla\phi(\mathbf{r},t) - \phi(\mathbf{r},t)\nabla\phi^*(\mathbf{r},t)\right)$$

$$\mathbf{j}(\mathbf{r},t) \equiv mn_c\mathbf{v}_s(\mathbf{r},t) \qquad \mathbf{v}_s(\mathbf{r},t) = \frac{\hbar}{m}\nabla\theta(\mathbf{r},t)$$

superfluid velocity - **not** the velocity of a particle

complex function is zero iff both its real and imaginary parts are zero:

Re = 0: continuity equation

$$\frac{\partial n_c}{\partial t} + \nabla\cdot(n_c\mathbf{v}_s) = 0$$

equivalent to solving GPE!

Im = 0: Euler-like equation

$$m\frac{\partial\mathbf{v}_s}{\partial t} = -\nabla\left(\frac{1}{2}m\mathbf{v}_s^2 + V_{ext}(\mathbf{r},t) + gn_c - \frac{\hbar^2}{2m\sqrt{n_c}}\nabla^2\sqrt{n_c}\right)$$

quantum hydrodynamics reduces to classical hydrodynamics

irrotational inviscid

classical

$$\frac{\partial \mathbf{v}}{\partial t} = \cancel{\mathbf{v} \times (\nabla \times \mathbf{v})} - \nabla\left(\frac{\mathbf{v}^2}{2}\right) - \frac{1}{m}\nabla U - \frac{1}{mn}\nabla p + \cancel{\nu\nabla^2\mathbf{v}}$$

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = -\mathbf{v}\times(\nabla\times\mathbf{v}) + \nabla\left(\frac{\mathbf{v}^2}{2}\right)$$

uniform

quantum

$$\frac{\partial \mathbf{v}}{\partial t} = \underbrace{\frac{1}{m}\nabla\left(\cancel{\frac{\hbar^2}{2m\sqrt{n_c}}\nabla^2\sqrt{n_c}}\right) - \nabla\left(\frac{\mathbf{v}^2}{2}\right)}_{\text{quantum pressure} \qquad \text{momentum}} - \frac{1}{m}\nabla V_{ext}(\mathbf{r},t) - \frac{1}{mn_c}\nabla p$$

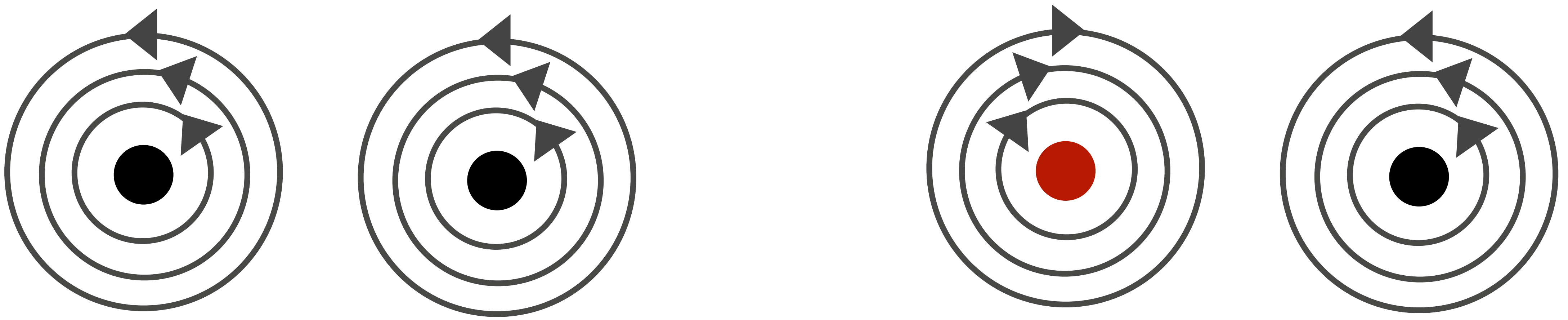
$$\mu = gn_c$$

$$dp = n_c d\mu$$

Gibbs-Duhem

superfluid velocity vs particle velocity!

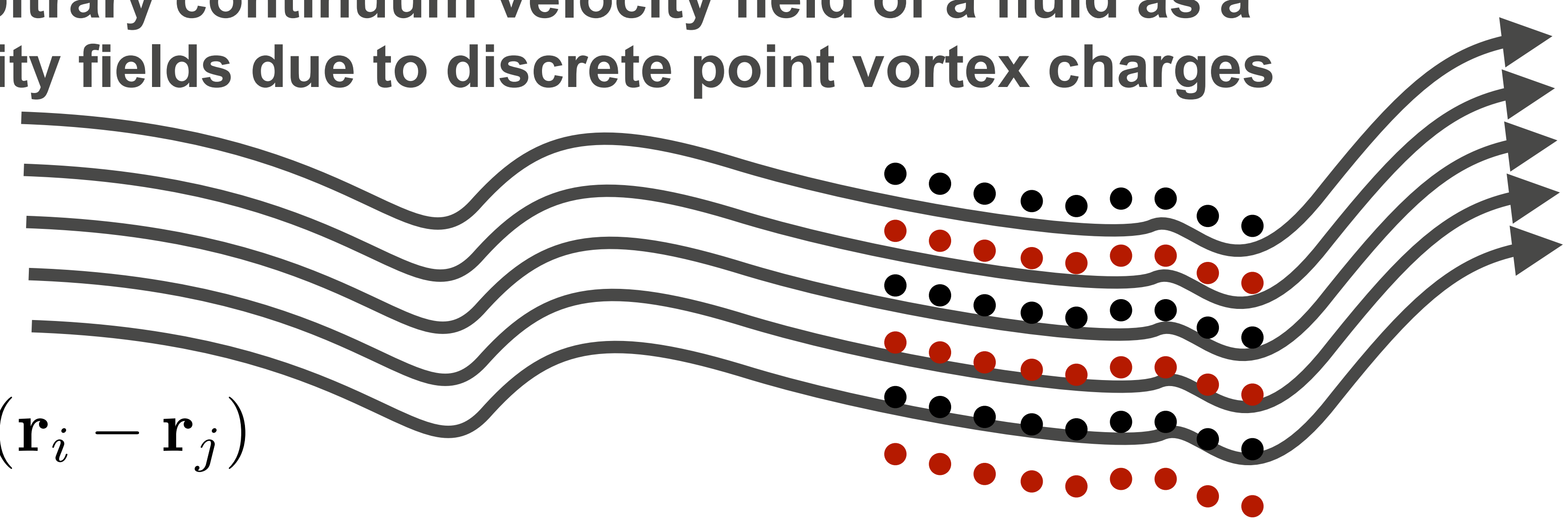
2D point vortex approximation: only consider kinetic energy of flow



$$E_v = \int \frac{1}{2} m|\phi|^2 v^2 d\mathbf{r} = \frac{mn}{2}\int_{r_c}^{R} \frac{h^2\ell^2}{m^2 r^2} r dr 2\pi L = \frac{\ell^2 n\hbar^2 \pi L}{m} \ln\left(\frac{R}{|\ell| r_c}\right)$$

$$E = \int \phi^*(\mathbf{r}) \left(-\frac{\hbar^2}{2m}\nabla^2\right) \phi(\mathbf{r}) dV$$

describe an arbitrary continuum velocity field of a fluid as a sum of N velocity fields due to discrete point vortex charges

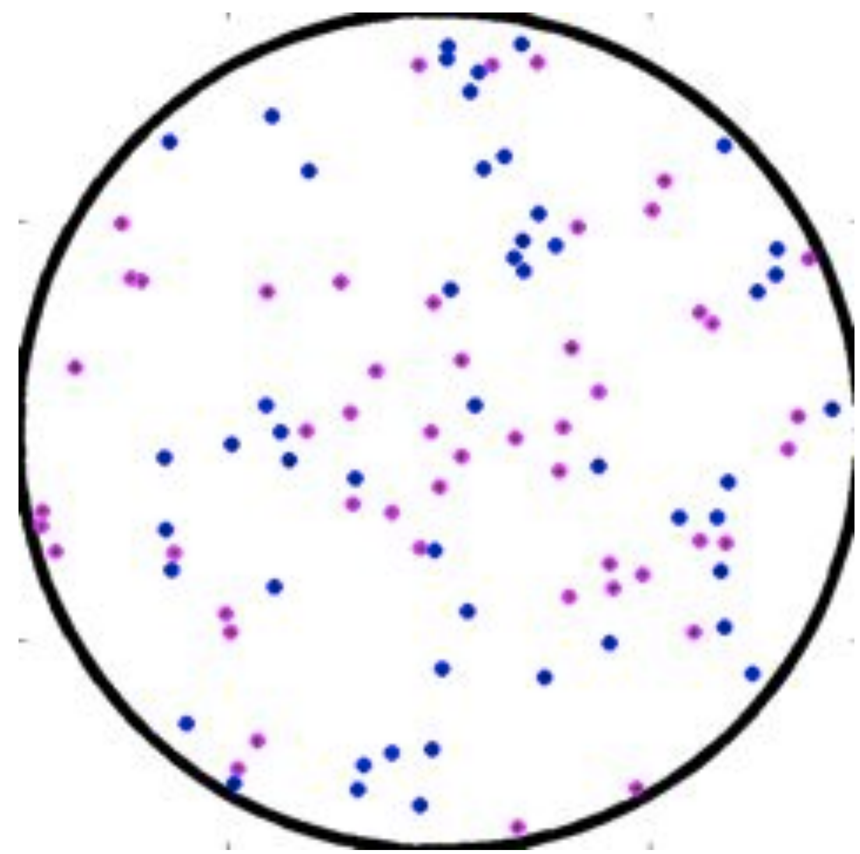


$$H = -\sum_{i<j} \Gamma_i \Gamma_j \log\left(\mathbf{r}_i - \mathbf{r}_j\right)$$

$$\mathbf{F}_{\mathrm{Magnus}} = \Gamma \hat{\mathbf{e}}_z \times (\mathbf{v} - \mathbf{v}')$$

2D point vortex approximation in a disc geometry

$$
\begin{aligned}
H_\circ &= H_{\text{vortices}} + H_{\text{images}} + H_{\text{self}}, \\
&= -\frac{\rho_s \kappa^2}{4\pi} \sum_{i<j} s_i s_j \log\left(r_{ij}^2\right) \\
&+ \frac{\rho_s \kappa^2}{4\pi} \sum_{i<j} s_i s_j \log\left(1 - 2(x_i x_j + y_i y_j) + |z_i|^2 |z_j|^2\right) \\
&+ \frac{\rho_s \kappa^2}{4\pi} \sum_{i} s_i^2 \log\left(1 - r_i^2\right),
\end{aligned}
$$



particle in 1D

$$\dot{p} = -\frac{\partial H}{\partial q}$$

$$\dot{q} = \frac{\partial H}{\partial p}$$

vortex in 2D

$$\dot{x} = -\frac{\partial H}{\partial y}$$

$$\dot{y} = \frac{\partial H}{\partial x}$$

phase space equals real space!

boundedness of phase space result in negative absolute temperature states (entropy has a maximum)!

two-dimensional systems are special

$$\nabla^2 U(\mathbf{r}) = -2\pi\delta(\mathbf{r}-\mathbf{r}')$$

3D $U \propto \frac{1}{r}$ **gravity, electromagnetism**

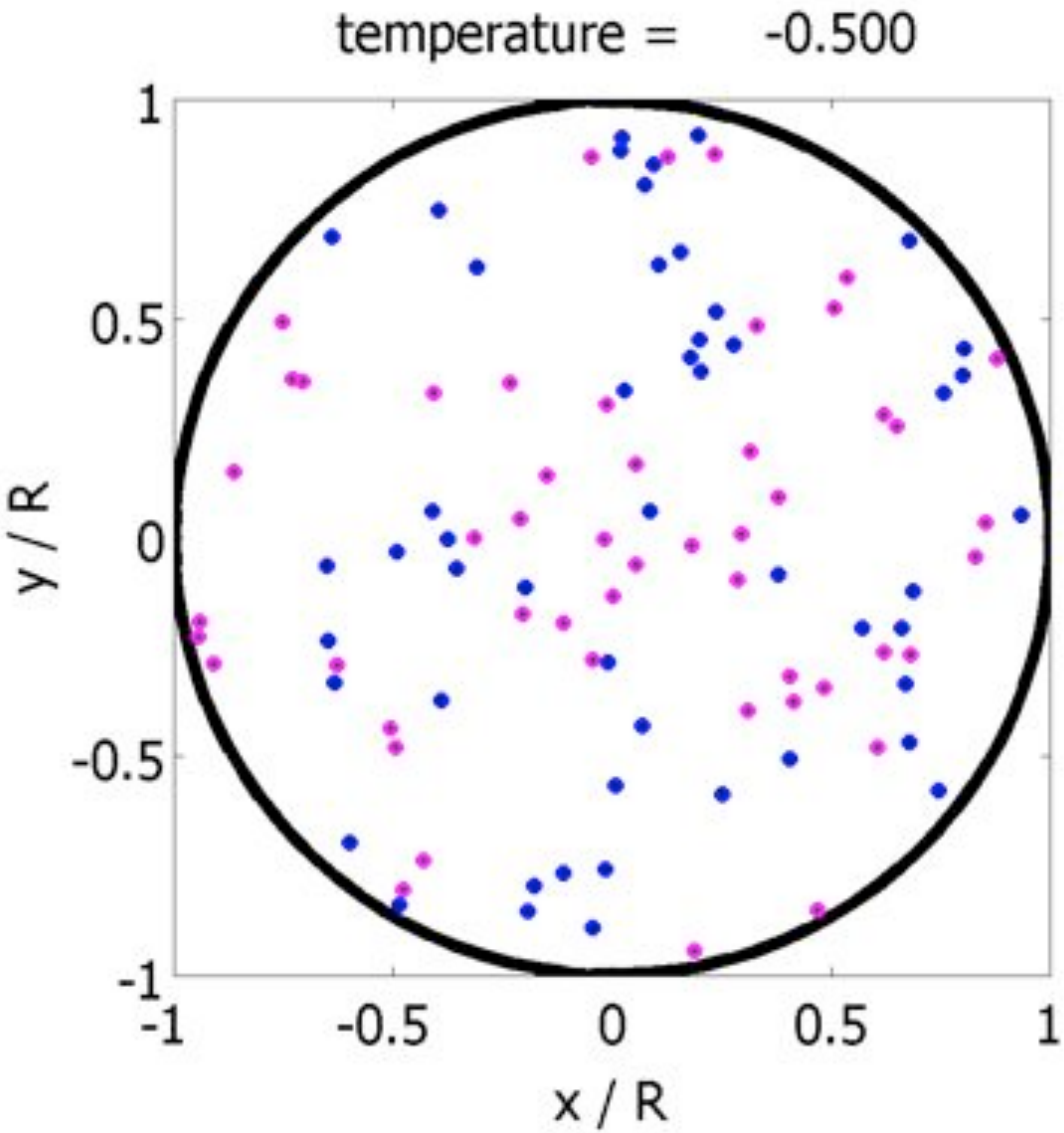
2D $U \propto \boxed{\log(r)}$ $S \propto 2\boxed{\log(r)}$

phase space equals real space: entropy

1D $U \propto r$

$$p(F) = e^{-\beta F} = e^{-\beta(E-TS)}$$

Monte Carlo thermodynamics



phase transitions in a two-dimensional Coulomb gas

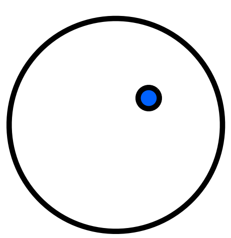
$$p(F) = e^{-\beta F} = e^{-\beta(E-TS)}$$

positive temperature

(low entropy, low energy)

$$E = \frac{\rho\kappa^2}{4\pi}\log\left(\frac{R}{\xi}\right)$$

$$S = 2k_{\mathrm{B}}\log\left(\frac{R}{\xi}\right)$$



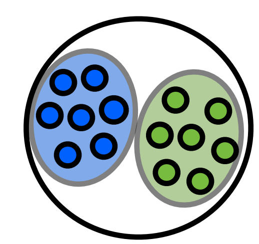
$$T_c = \frac{1}{2}\frac{\rho\kappa^2}{4\pi}$$

negative temperature

(low entropy, high energy)

$$E = 2\frac{\rho(\frac{N}{2}\kappa)^2}{4\pi}\log\left(\frac{D}{\xi}\right)$$

$$S = \frac{N}{2}k_{\mathrm{B}}\log\left(\frac{\xi_{\mathrm{OV}}}{R}\right)$$



$$T_c = -\frac{\rho\kappa^2}{8\pi}N$$

negative temperature Onsager vortex phase transition

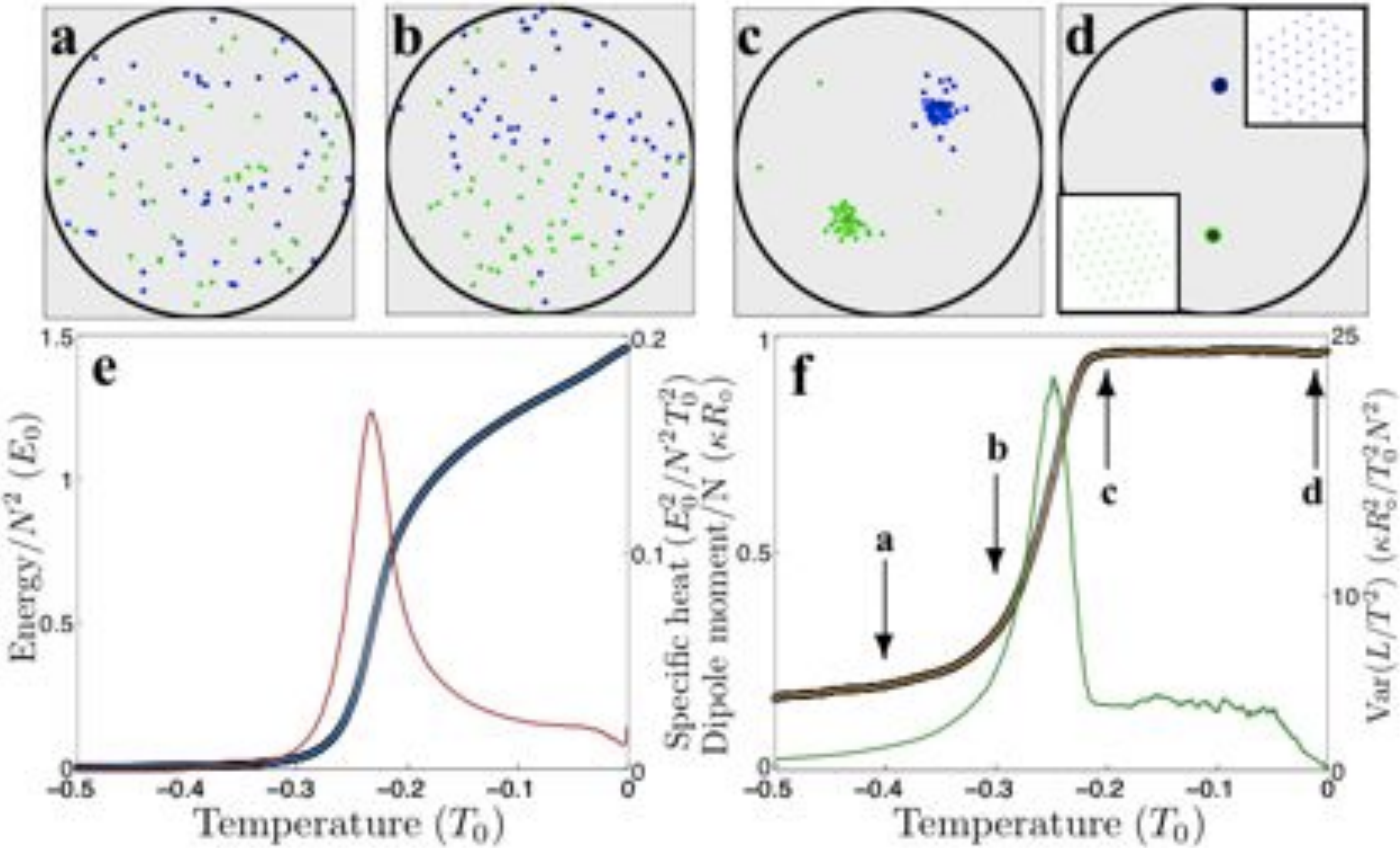
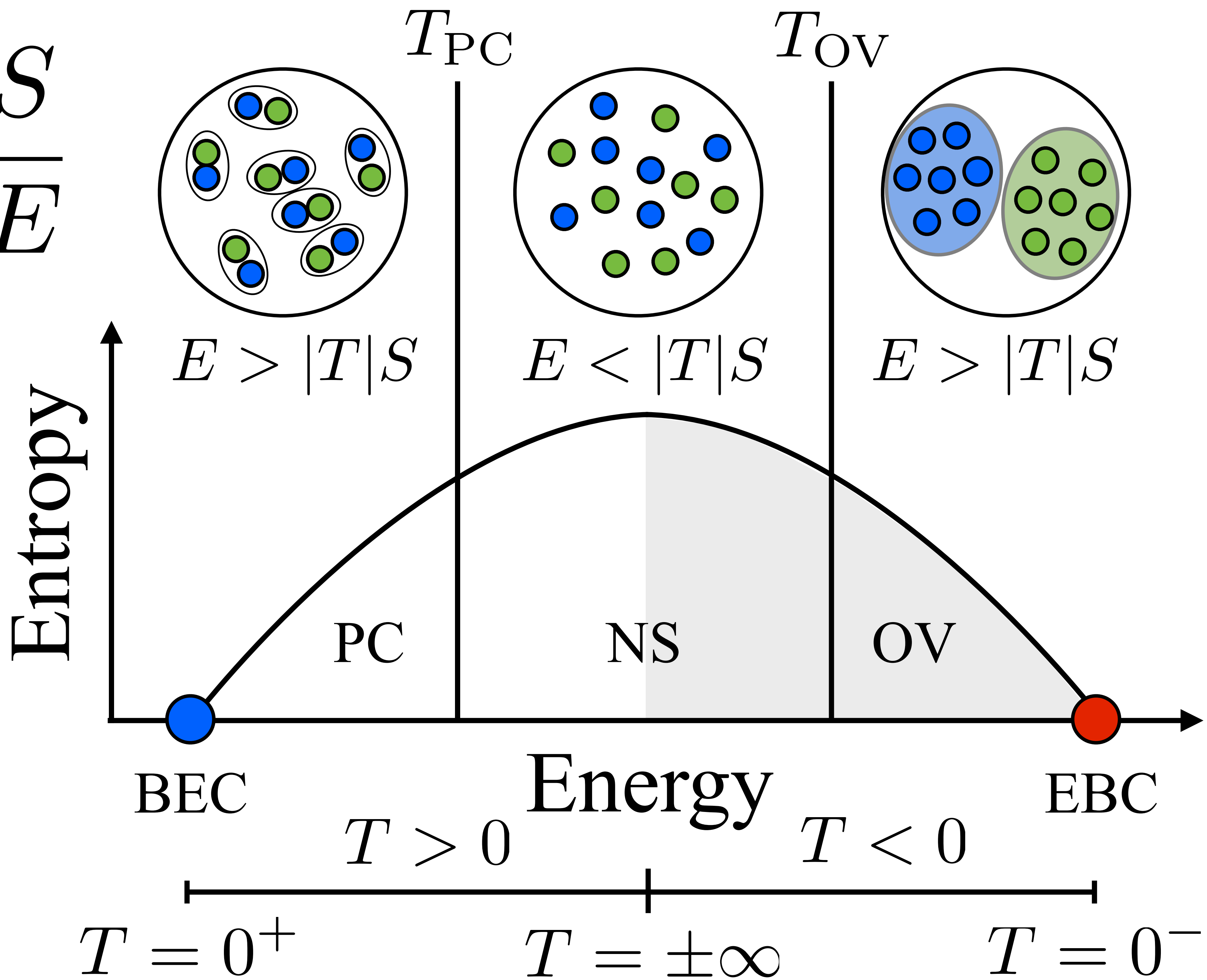


diagram of phases

$$\frac{1}{T} = \frac{\partial S}{\partial E}$$



So much for the point vortex model.

What about real superfluids?

$$i\hbar\partial_t\psi(\mathbf{r}) = \left[-\frac{\hbar^2}{2m}\nabla^2 + U_{\mathrm{disc}}(\mathbf{r}) + g|\psi(\mathbf{r})|^2\right]\psi(\mathbf{r})$$

note: this GPE is conservative (no damping / dissipation)

EMERGENCE OF ORDER FROM TURBULENCE IN AN ISOLATED PLANAR SUPERFLUID

GROSS-PITAEVSKII DYNAMICS

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INITIAL NUMBER OF VORTICES: $N = 80$

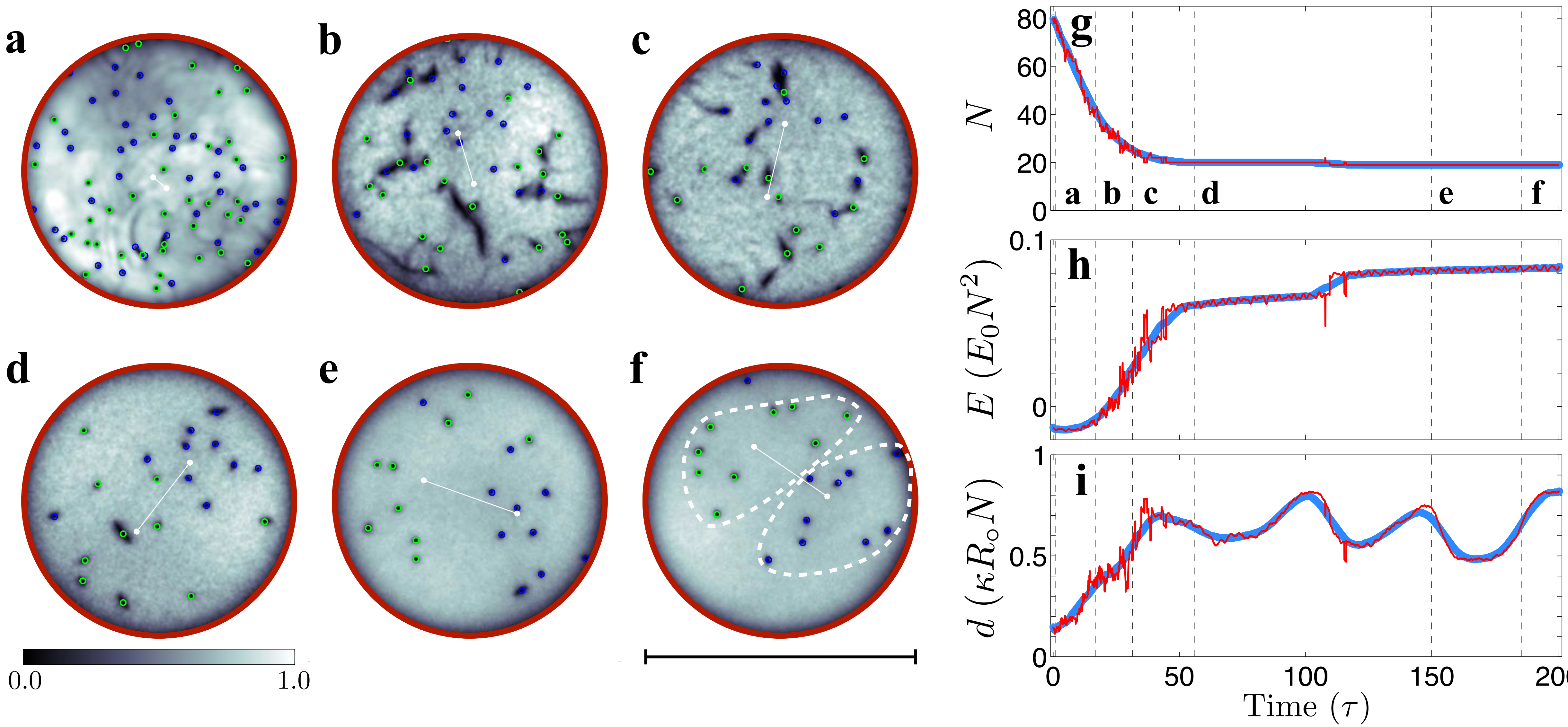
SYSTEM RADIUS: $R_\circ = 28\ a_{\rm osc}$

AXIAL THOMAS-FERMI LENGTH: $R_z = 4.3\ a_{\rm osc}$

CHEMICAL POTENTIAL: $\mu = 9.3\ \hbar\omega_{\rm osc}$

MONASH University

THE UNIVERSITY OF QUEENSLAND AUSTRALIA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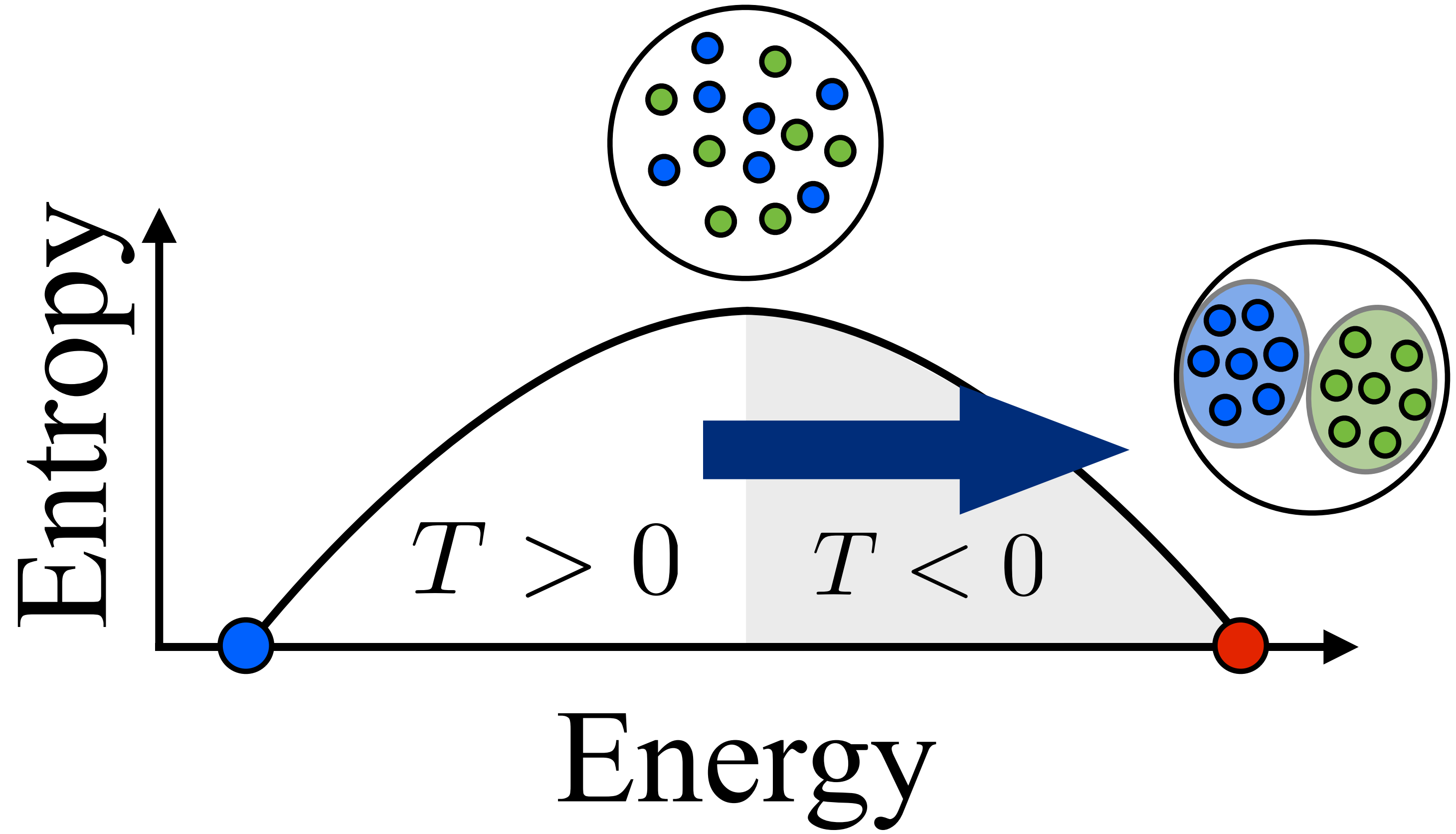
f
0.0
1.0
g
h
i
N
$E\ (E_0 N^2)$
$d\ (\kappa R_\circ N)$
Time (τ)
80
60
40
20
0
0.1
0
1
0.5
0
50
100
150
200



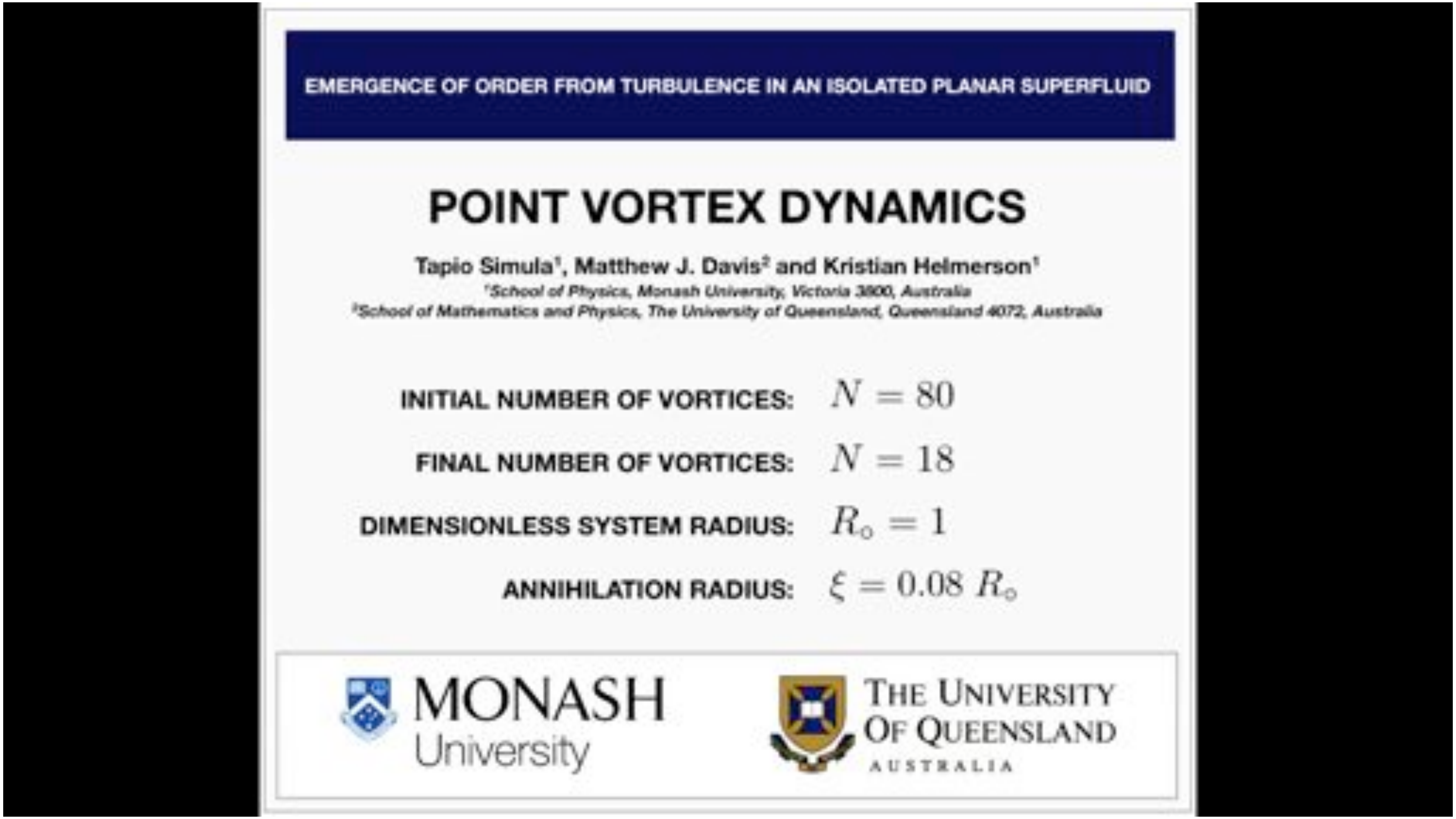
annihilating vortices are "cold", leaving remaining vortices to equilibriate at a higher mean energy per vortex!

second law and arrow of time?

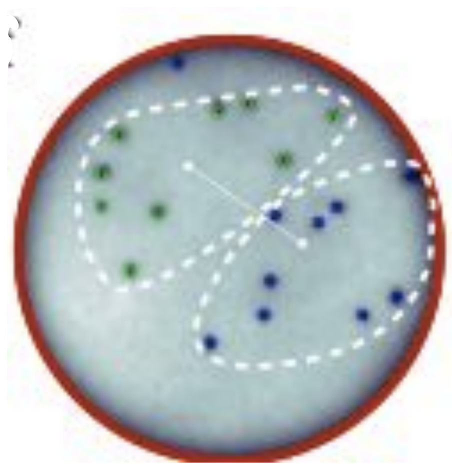
two degrees of freedom!

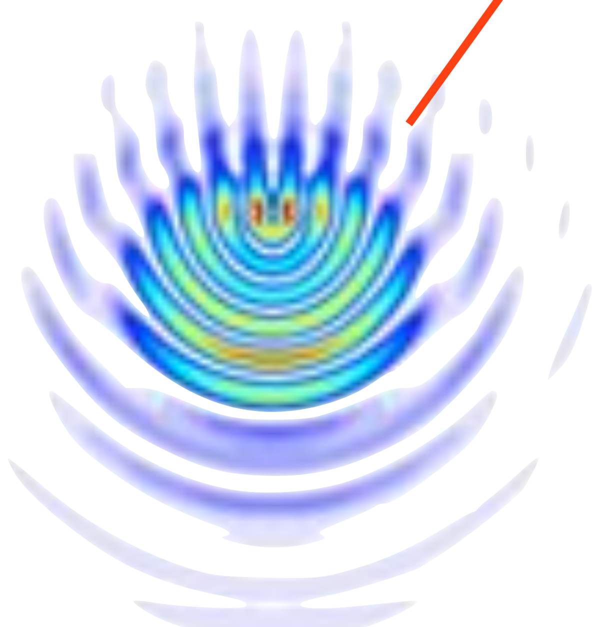
vortices and sound waves

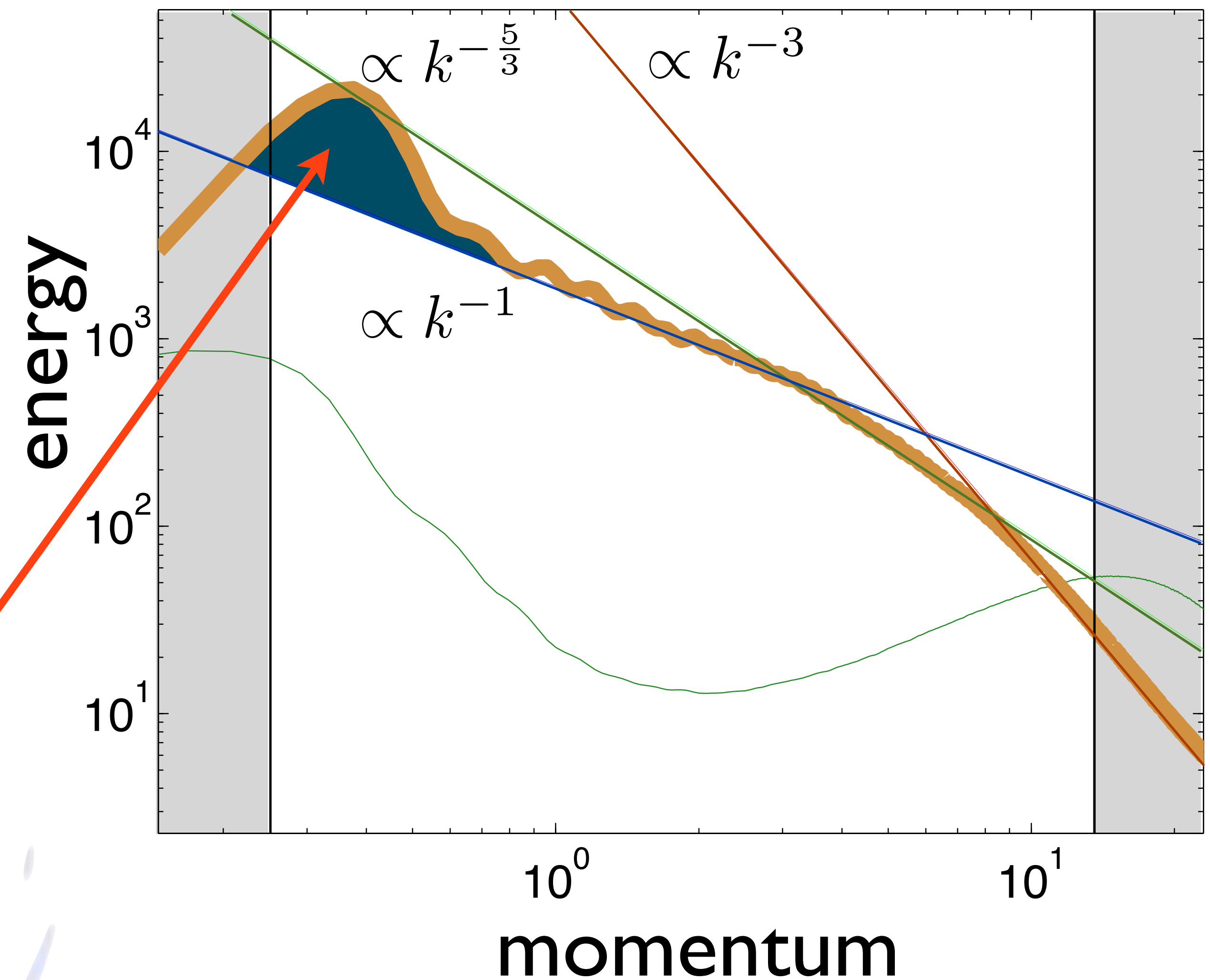
evaporative heating of vortices

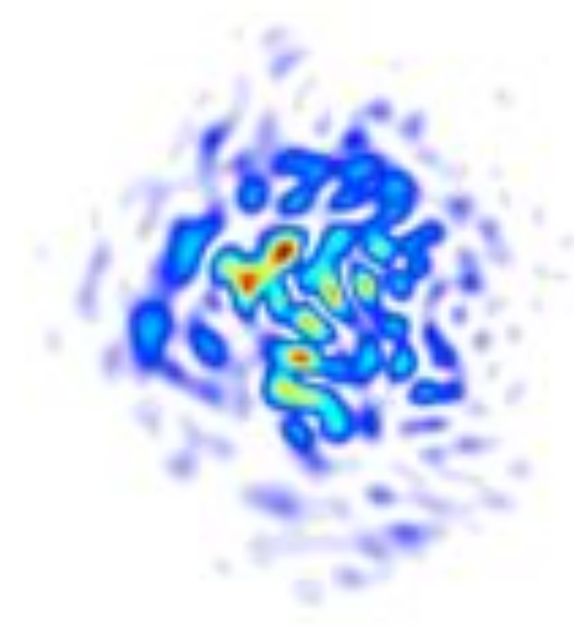


spectral condensation & Onsager vortices via evaporative heating









Einstein-Bose condensate

thermal state

2D analogies

- **GPE - Navier-Stokes correspondence**
  - **quantum hydrodynamics**
- **XY model 2D Maxwell correspondence**
  - **massless vortex charges**
- **gravity - fluid correspondence**
  - **holographic principle**

Gross-Pitaevskii equation --- Navier-Stokes

irrotational

inviscid

classical

$$\frac{\partial \mathbf{v}}{\partial t} = \cancel{\mathbf{v} \times (\nabla \times \mathbf{v})} - \nabla\left(\frac{\mathbf{v}^2}{2}\right) - \frac{1}{m}\nabla U - \frac{1}{mn}\nabla p + \cancel{\nu\nabla^2\mathbf{v}}$$

recovered by
finite T effects

uniform

quantum

$$\frac{\partial \mathbf{v}}{\partial t} = \underbrace{\cancel{\frac{1}{m}\nabla\left(\frac{\hbar^2}{2m\sqrt{n_c}}\nabla^2\sqrt{n_c}\right)} - \nabla\left(\frac{\mathbf{v}^2}{2}\right)}_{\text{quantum pressure} \qquad \text{momentum}} - \frac{1}{m}\nabla V_{ext}(\mathbf{r},t) - \frac{1}{mn_c}\nabla p$$

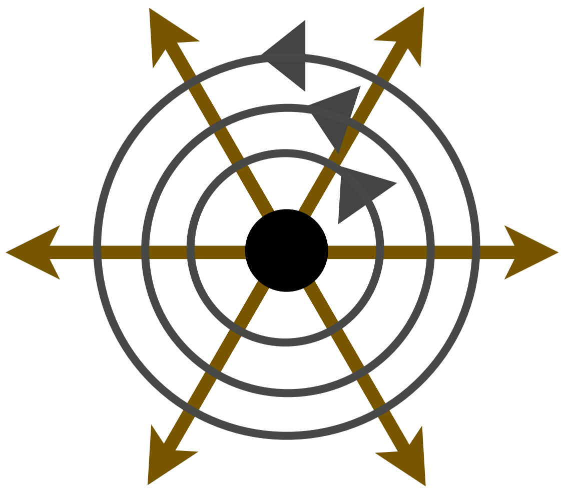
XY model equivalence to 2D electromagnetism (Maxwell)

Coulomb charge (electric charge) **vortex charge (quantum of circulation)**

$$H = -J_0 \sum_{\langle i,j \rangle} \cos(\phi_i - \phi_j)$$

$$\mathbf{E}_v \equiv \alpha(\nabla\phi \times \mathbf{e}_z)$$

$$\mathbf{B}_v \equiv \beta \, \frac{d\phi}{dt} \, \mathbf{e}_z$$



$$\nabla \cdot \mathbf{B}_v = 0 \qquad \nabla \times \mathbf{E}_v = -\frac{1}{c_o}\frac{\partial \mathbf{B}_v}{\partial t}$$

$$\nabla \cdot \mathbf{E}_v = 2\pi\rho \quad \nabla \times \mathbf{B}_v = \frac{1}{c_0}\frac{\partial \mathbf{E}_v}{\partial t} + \frac{2\pi}{c_0}\mathbf{j}$$

Holographic Vortex Liquids and Superfluid Turbulence
Science 341, 368 (2013)

Holographic Path to the Turbulent Side of Gravity
Phys. Rev. X 4, 011001 (2014)

further directions

- spinor (vector) order parameters

$$\psi = (\phi_1, \phi_2, \phi_3, \ldots) \quad L, S$$

- fractional vortices, (monopoles, skyrmions...)

$$\psi = \left(e^{i\theta}, 1\right) \qquad \frac{1}{2}, \frac{1}{2}$$

- spin-turbulence

$$E(k) \propto k^{-\frac{7}{3}}$$

- non-Abelian vortices (topological QC ...)

$$\psi = \left(e^{i\theta}, 0, 0, 1, 0\right)$$

- "rung-mediated" non-Abelian turbulence
    - vortex reconnections forbidden!

take home slide on 2D (quantum) turbulence

- enstrophy conservation leads to an inverse energy cascade
- finite phase space leads to absolute negative temperatures
- emergence of giant “Onsager vortices” via inverse energy cascade or vortex evaporative heating mechanism