Condensed Matter Emulation

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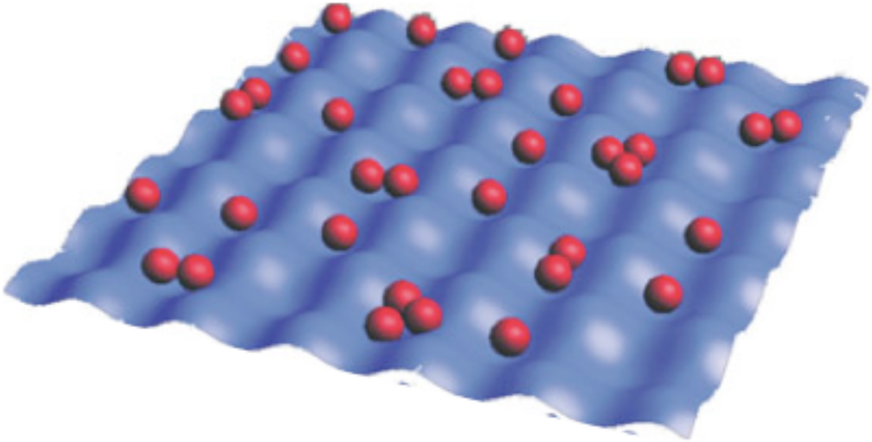
Condensed Matter

- Disordered
- Unknown interactions
- Little control



Cold Atoms

- Tunable dispersion
- Tunable interactions
- “Perfect” control
- Clean or controlled disorder
- Engineered Hamiltonians

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These 2 Lectures

- Lecture 1:
  - Introduction to emulation
  - The integer/fractional quantum Hall effect (solid state)
  - Emulation of the quantum Hall effect (ultra-cold atoms)

- Lecture 2:
  - Coupled Atom Cavity (CAC) systems
  - Bose-Hubbard model (ultra-cold gases and CAC systems)
  - Fractional Quantum Hall Effect (CAC systems)
  - Supersolids (ultra-cold gases and CAC systems)

Reference Material

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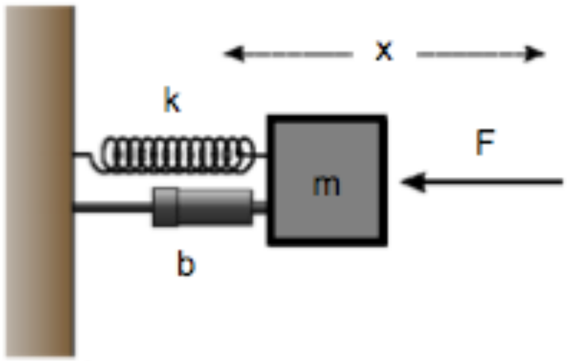
- Condensed matter emulation
  - *Ultracold atomic gases in optical lattices: mimicking condensed matter physics and beyond*, M. Lewenstein, A. Sanpera, V. Ahufinger, B. Damski, A. Sen and U. Sen, Advances in Physics **56**, 243 (2007)
  - *Many-body physics with ultracold gases*, I. Bloch, J. Dalibard and W. Zwerger, Rev. Mod. Phys. **80**, 885 (2008)
- Integer/Fractional Quantum Hall effect
  - *Introduction to the fractional quantum Hall effect*, S. M. Girvin, http://www.bourbaphy.fr/girvin.ps
  - *Rotating trapped Bose-Einstein condensates*, A. L. Fetter, Rev. Mod. Phys. **81**, 647 (2009)

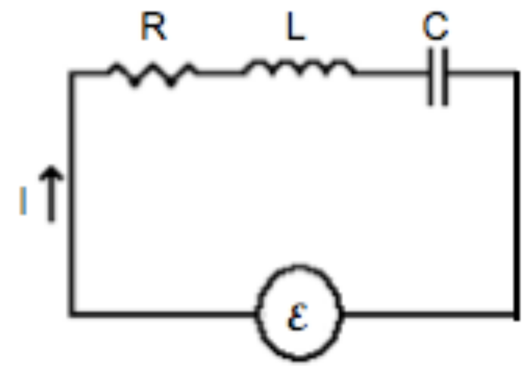
Equivalence of Physical Systems

Driven damped oscillator



$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F\cos(\omega t)$$

RLC circuit



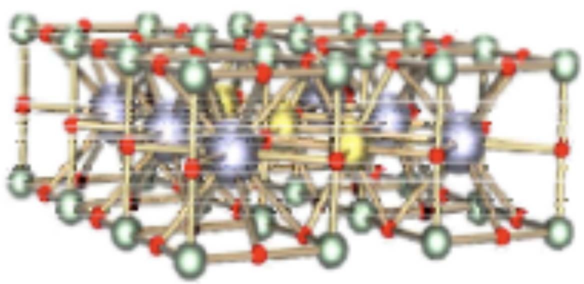
$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}x = \varepsilon\cos(\omega t)$$

ANALOGOUS MECHANICAL & ELECTRICAL QUANTITIES

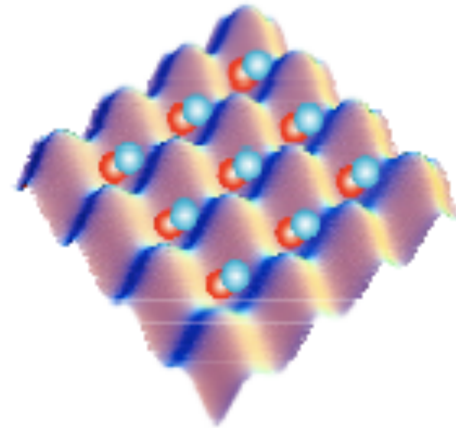
Mechanical		Electrical	
x	Displacement	q	Charge
$\dot{x}$ (v)	Velocity	$\dot{q}$ (I)	Current
m	Mass	L	Inductance
b	Friction	R	Resistance
1/k	Mechanical Compliance	C	Capacitance
F	Amplitude of impressed force	$\mathcal{E}$	Amplitude of impressed emf

Equivalence of Physical Systems

YBCO superconductor



Optical Lattice

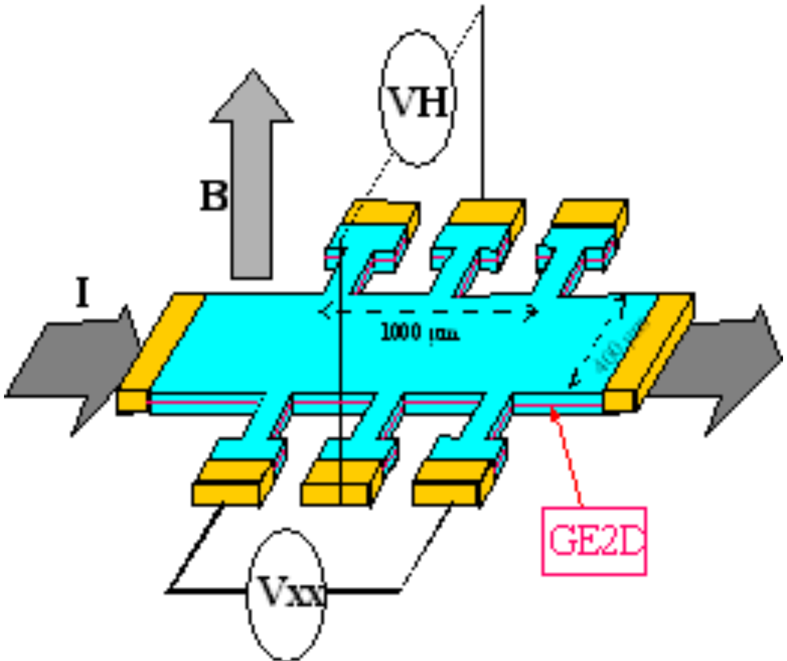


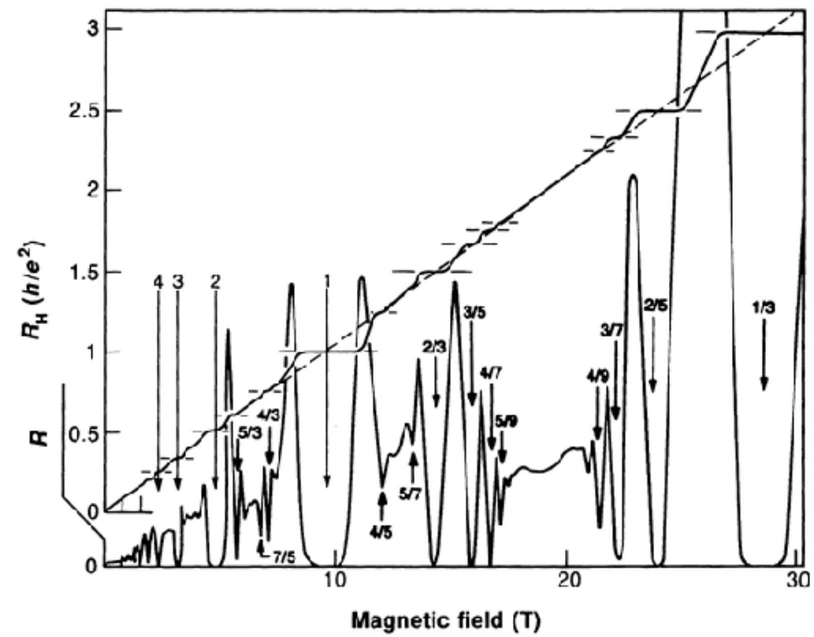
$$H = -t_{e(a)} \sum_{\langle i,j \rangle,\sigma} \left(c^{\dagger}_{i,\sigma} c_{j,\sigma} + \text{h.c.} \right) + U_{C(s)} \sum_{i} n_{i,+1} n_{i,-1}$$

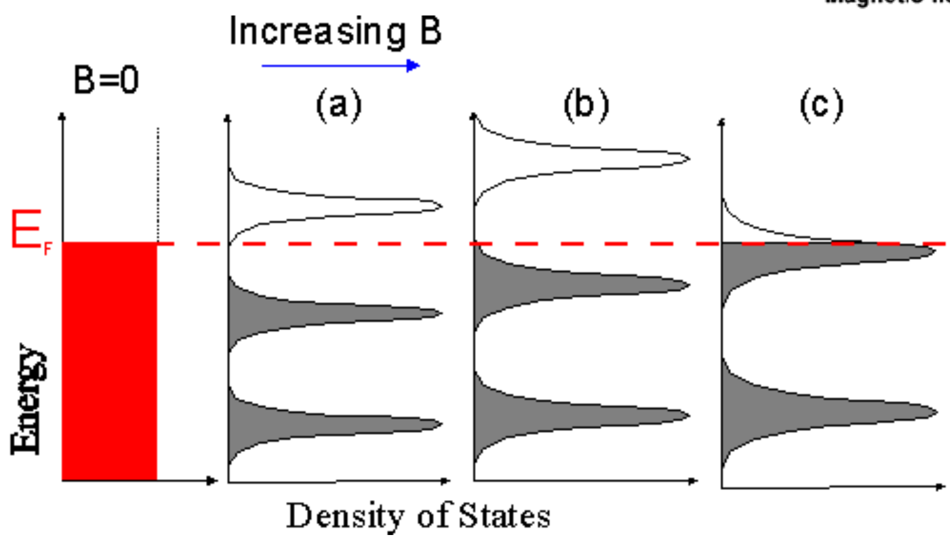
ANALOGOUS CONDENSED MATTER AND OPTICAL LATTICE QUANTITIES

	Condensed Matter		Atom-Optical
Carriers	Electron/Holes		Fermionic atoms
e^-e^-	Coulomb charge coupling	s	S-wave scattering length
m_e	Electron mass	m_a	Atomic mass
U_c	Coulomb Interaction	U_s	S-wave Interaction
t_e	Electronic tunneling energy	t_a	Atomic tunneling energy
Lattice	Atomic ions		Optical standing waves
a, b, c	Lattice Constants	$(\lambda_x, \lambda_y, \lambda_z)/2$	Optical wavelength
V_{ion}	Binding energy	V_{lat}	Lattice depth

The Quantum Hall Effect







The Hall Effect

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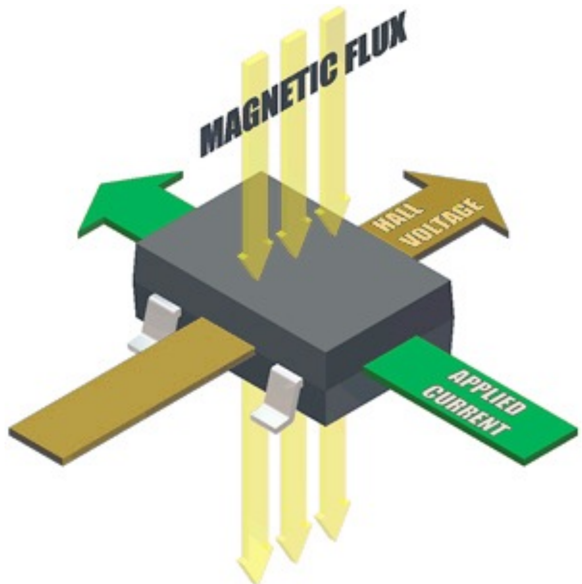
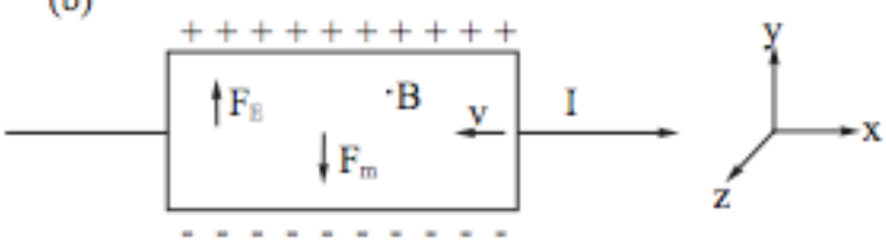


$$\mathbf{J} = -ne\mathbf{v}$$

(b)



Force on carrier

$$\mathbf{F} = -e\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) = \mathbf{F_E} + \mathbf{F_M}$$

Equilibrium

$$\mathbf{F_E} = -\mathbf{F_M}$$

y-component

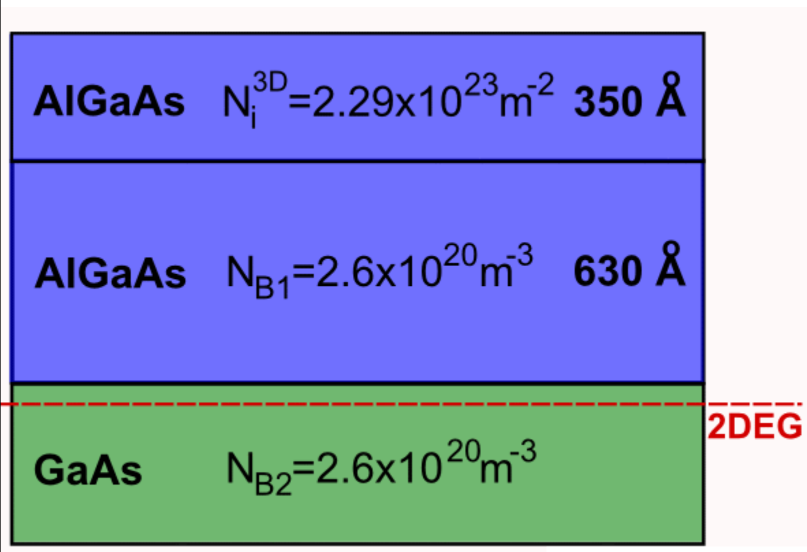
$$F_y = ev_x B_z - eE_y = 0$$

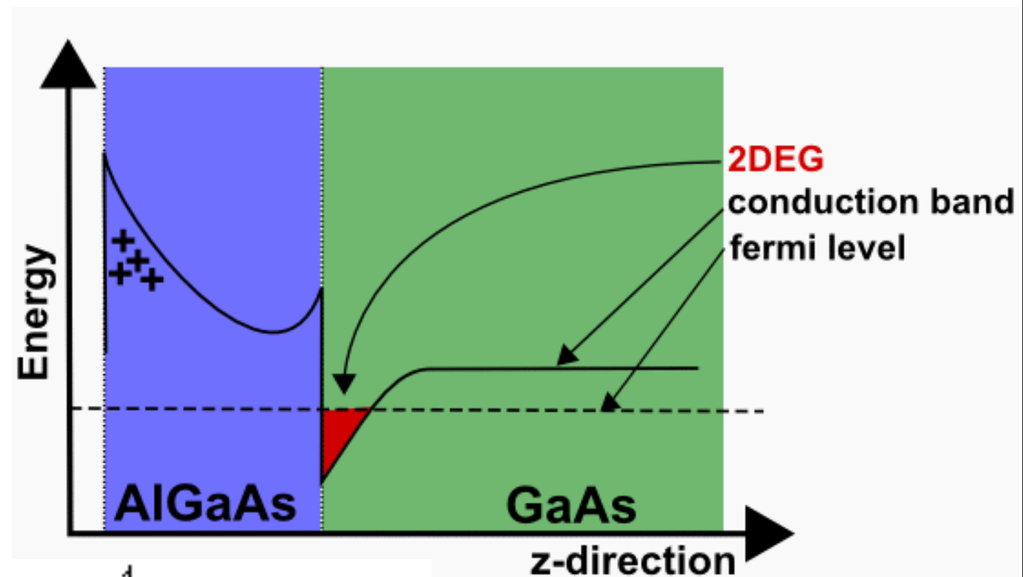
$$E_y = -\frac{J_x B_z}{ne}$$

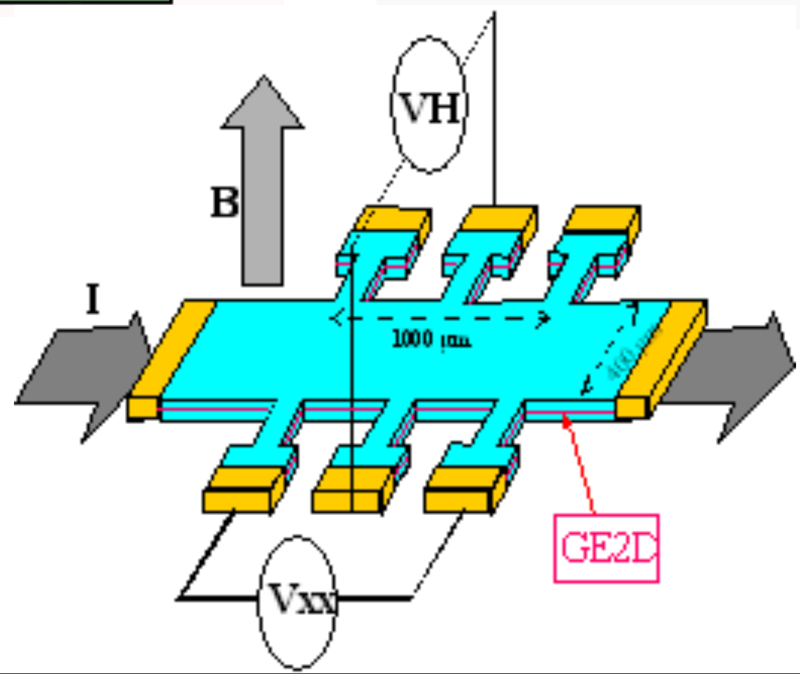
Hall coefficient

$$R_H = \frac{E_y}{J_x B_z} = -\frac{1}{ne}$$

Experimental Setup (QHE)

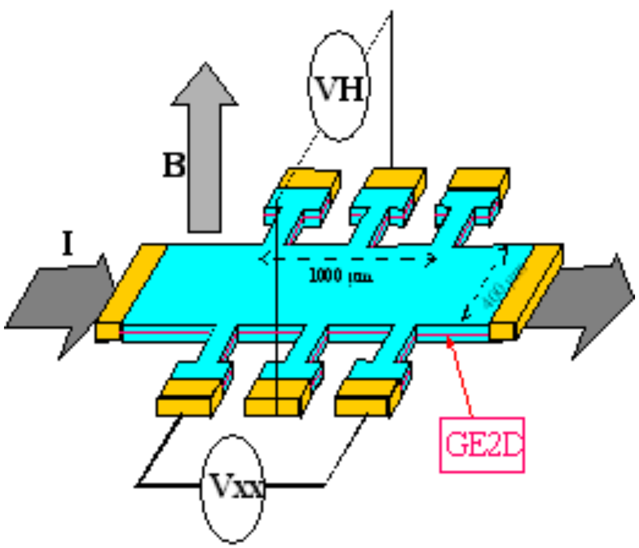


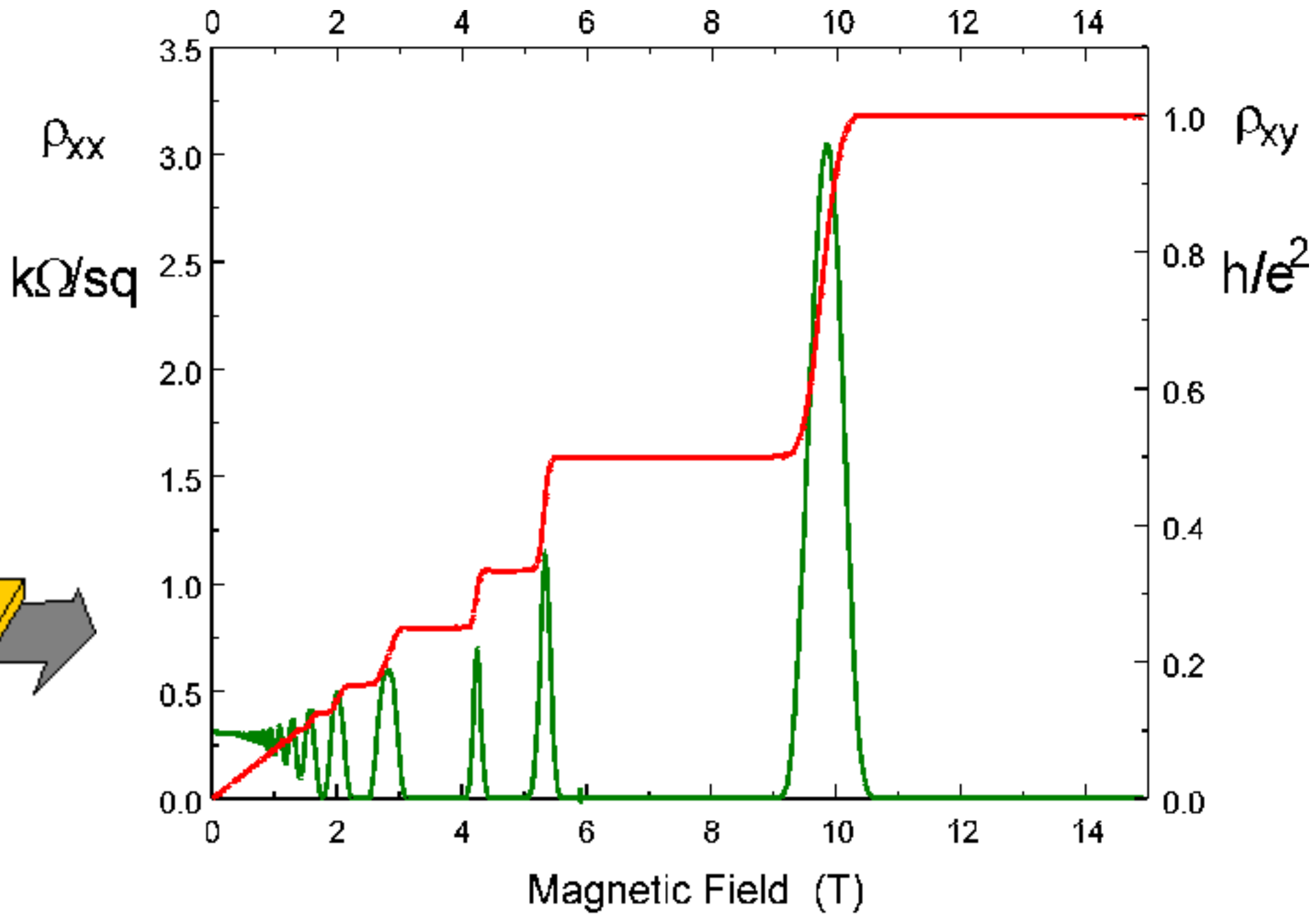




Experimental Observation (IQHE)

$$\frac{1}{\rho_{xy}} = \sigma_{xy} = \nu \frac{e^2}{h}, \;\; \nu = 1, 2, 3 \ldots \;\; \text{(Integer)}$$



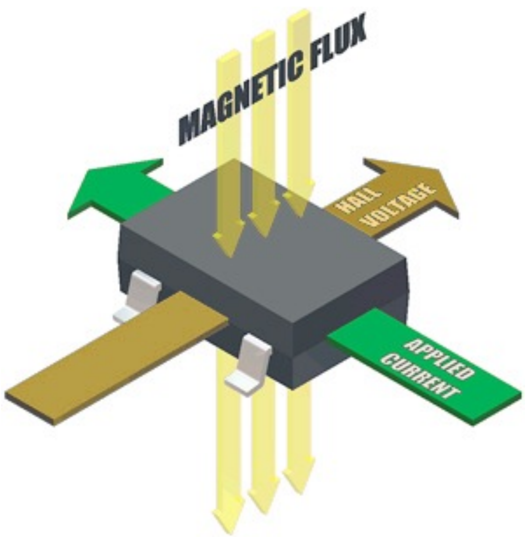


Quantum Treatment

Schrodinger Equation

$$\left[\frac{1}{2m}\left(i\hbar\nabla - q\mathbf{A}(\mathbf{r},t)\right)^2 + q\phi(\mathbf{r},t)\right]\psi(\mathbf{r},t) = i\hbar\frac{d}{dt}\psi(\mathbf{r},t)$$

$$\mathbf{B} = \nabla\times\mathbf{A} = B_z\hat{k}$$



Choice of gauge

$$\mathbf{A} = \hat{j}B_z x \quad \text{(Landau gauge)}$$

$$\mathbf{A} = -\hat{i}B_z y/2 + \hat{j}B_z x/2 \quad \text{(Symmetric gauge)}$$

Physical results independent of gauge

We choose Landau gauge

Landau Gauge

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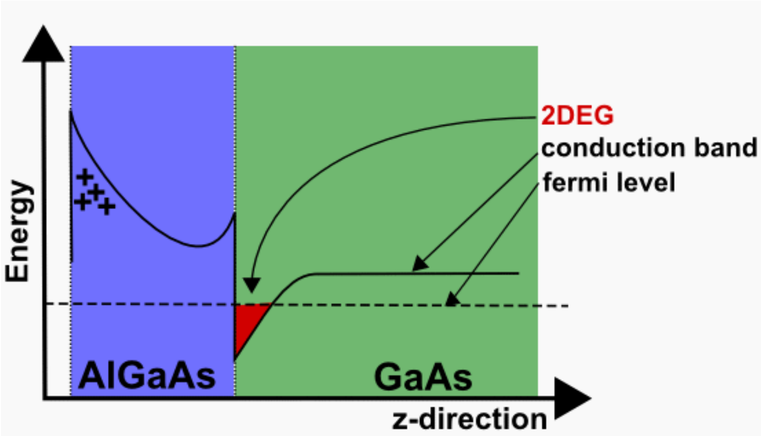
Schrodinger Equation (Landau gauge)

$$\left[-\frac{\hbar^2}{2m}\nabla^2 - \frac{ie\hbar B_z}{m}\frac{\partial}{\partial y} + \frac{(eB_z x)^2}{2m} + V(z)\right]\psi(\mathbf{r}) = E\psi(\mathbf{r})$$



Drop z-dependence (2DEG)

↓

Vector potential independent of y

↓

Plane wave solutions for y-direction: 1D Schrodinger Equation

1D Schrodinger Equation

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$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{1}{2}m\omega_c^2\left(x+\frac{\hbar k}{eB}\right)^2\right]u(x)=\epsilon u(x)$$

Schrodinger Equation for 1D harmonic oscillator

Vertex of parabolic potential displaced by $-\hbar k/eB$

Energy eigenvalues

$$\epsilon_{nk}=(n-1/2)\hbar\omega_c, \quad \text{where } n=1,2,3,...$$

Wavefunction

$$\psi_{nk}(x,y)\propto H_{n-1}\left(\frac{x-x_k}{l_b}\right)e^{-\frac{(x-x_k)^2}{2l_b^2}}e^{iky}, \quad \text{where } l_b=\sqrt{\hbar/|eB_z|}$$

The eigenvalues (Landau levels) depend on *n* but not *k*

Landau Levels DOS

B=0: 2D electron DOS is a constant

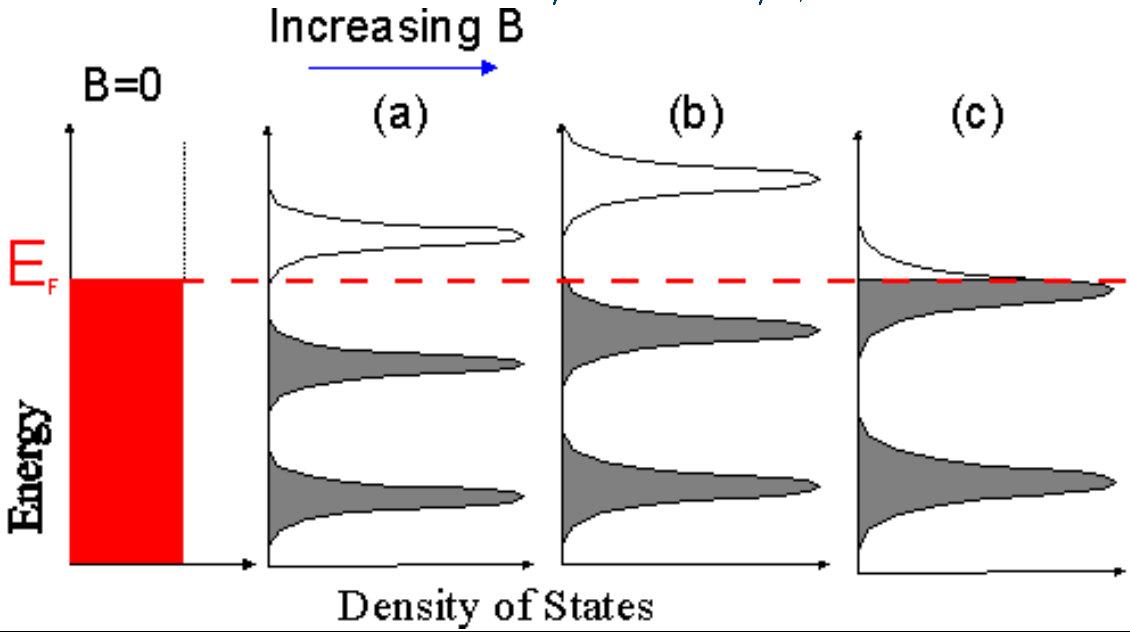
|B|>0: 2D electron DOS is a series of delta functions

↓

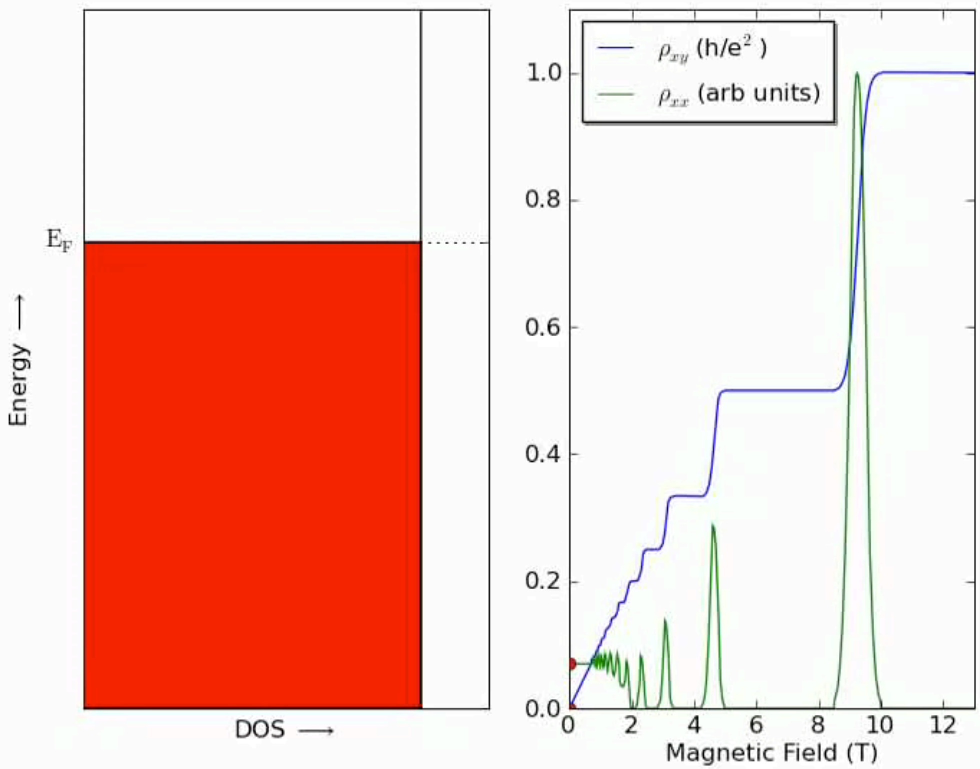
Landau Levels (LLs)

Number of states in each LL per unit area

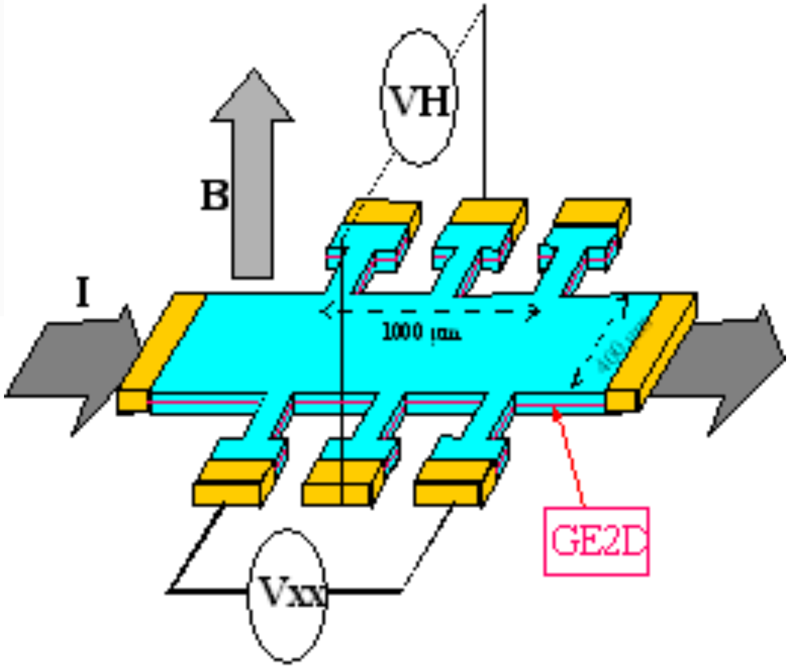
$$n_B = eB/h = B/\phi_0$$



IQHE



E_F
Energy ⟶
DOS ⟶
ρ_xy (h/e²)
ρ_xx (arb units)
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
12
Magnetic Field (T)



VH
B
I
1000 μm
Vxx
GE2D

Landau Levels: Transport

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Number of occupied LLs

$$\nu = \frac{n_{2D}}{n_B} = 2\pi l_b^2 n_{2D}$$

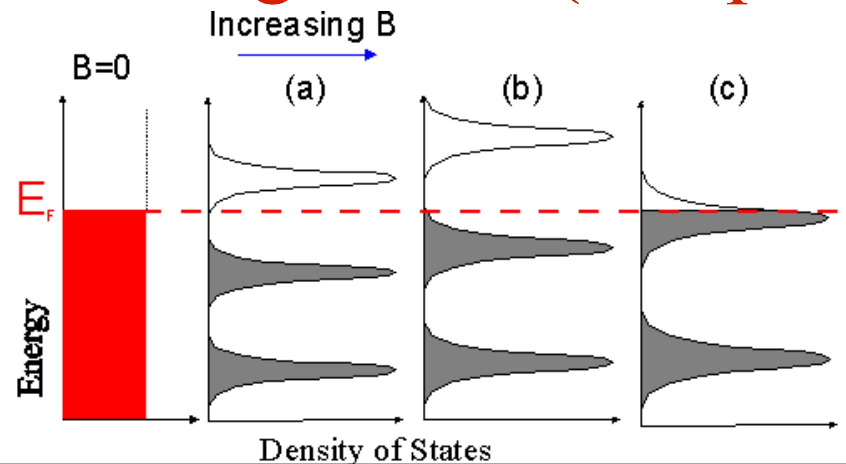
Between LLs (n integer)

$$B_n = \frac{h n_{2D}}{e n}$$

↓

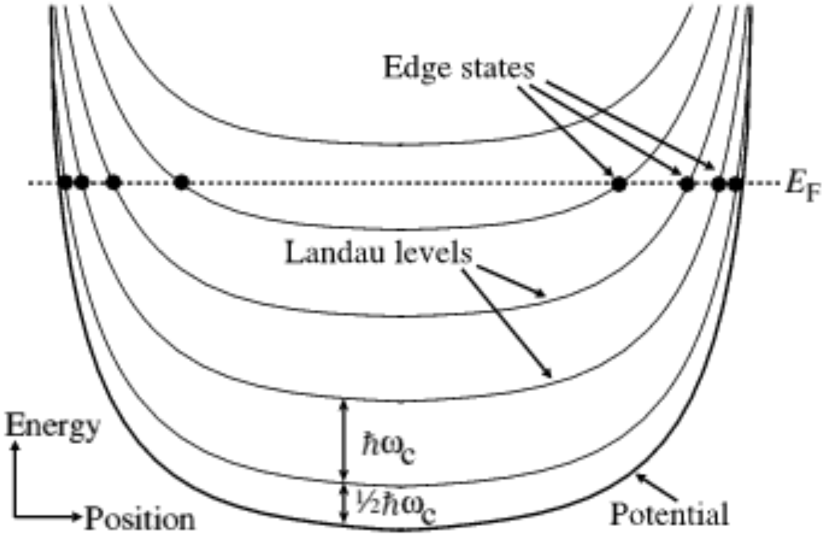
Fermi energy between LLs: low DOS (incompressible)

Within LL high DOS (compressible)



Confinement

LLs: confined geometry



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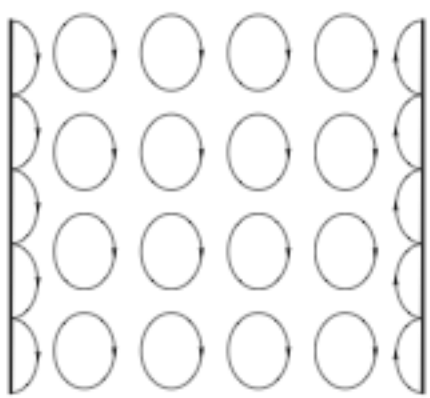
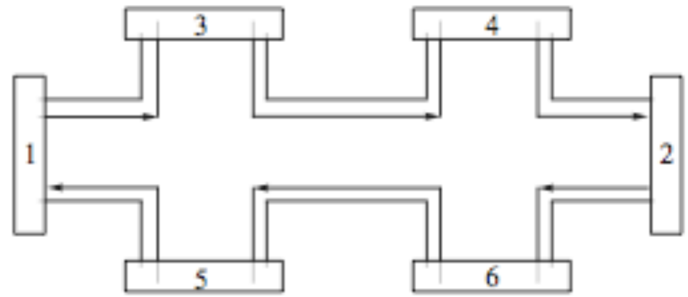


Hall bar schematic



Fermi energy between LLs

↓

Edge state transport

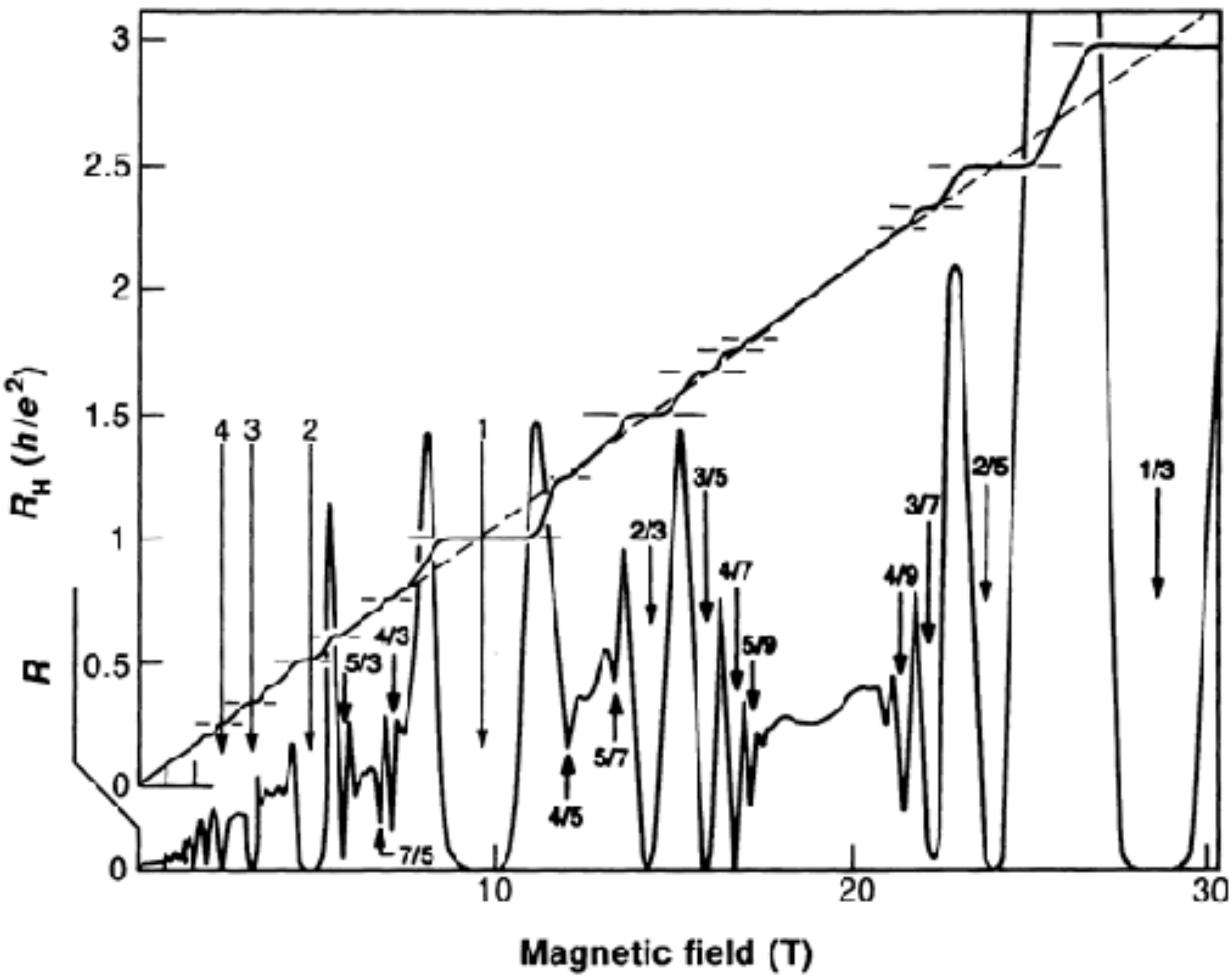
$$I = -\frac{Ne^2}{h}V$$

$$\rho_{xy} = \frac{V_5 - V_3}{I} = -\frac{V_1}{I} = \frac{h}{Ne^2}$$

Assumptions: No disorder

- Scattering between edge states in the same edge
  - Is forward hence no effect (exception high currents)
- Scattering between opposing edges
  - Very very weak if Fermi energy is between Landau levels
- Surely Fermi energy adjusts to always be in a LL: Why are plateaus wide?
  - Disorder important: localized states between LLs in bulk
  - Finite DOS between LLs
  - Do not contribute to electrical properties (localized)

The Surprise (FQHE)



Theoretical Model of FQHE

- Controlled by Coulomb repulsion between electrons
  - Ignore disorder
  - Discover the nature of the *special* many-body correlated state
- Consider symmetric gauge (remember results are gauge independent)

$$\mathbf{A} = -\frac{1}{2}\mathbf{r} \times \mathbf{B}$$

  - Preserves rotational symmetry
- Consider only Lowest Landau Level (LLL): No interactions

$$\phi_m = \frac{1}{\sqrt{2\pi l_b^2 2^m m!}} z^m e^{-\frac{1}{4}|z|^2}, \quad \text{where } z = (x+iy)/l_b$$

- All the states are degenerate: can have any linear combination

$$\Psi(x,y) = f(z) e^{-\frac{1}{4}|z|^2} \qquad f(z) = \prod_{j=1}^{N} (z - Z_j)$$

The LLL Many-Body State

$$\psi[z] = f[z] e^{-\frac{1}{4}\sum_j |z_j|^2}$$

f is a polynomial representing the Slater determinant with all states occupied

2 particles

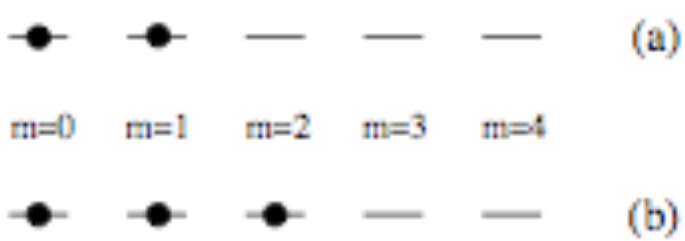
$$f[z] = \begin{vmatrix} (z_1)^0 & (z_2)^0 \\ (z_1)^1 & (z_2)^1 \end{vmatrix} = (z_1)^0 (z_2)^1 - (z_2)^0 (z_1)^1 = (z_2 - z_1)$$

3 particles

$$f[z] = \begin{vmatrix} (z_1)^0 & (z_2)^0 & (z_3)^0 \\ (z_1)^1 & (z_2)^1 & (z_3)^1 \\ (z_1)^2 & (z_2)^2 & (z_3)^2 \end{vmatrix} = -\prod_{i<j}^{3} (z_i - z_j)$$

N particles

$$f_N[z] = \prod_{i<j}^{N} (z_i - z_j)$$



Lauglin Variational Ansatz

$$f_N^m[z] = \prod_{i<j}^{N} (z_i - z_j)^m \qquad \nu = 1/m$$

To be analytic *m* must be an integer

To preserve antisymmetry *m* must be odd

$$\nu = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$$

In the plasma analogy the electron density is

$$n = \frac{1}{m} \frac{1}{2\pi l_b^2}$$

Other wave-functions developed to describe more general states in the hierarchy of rational filling factors at which quantized Hall plateaus were observed

Plasma Analogy (I)

$$|\Psi[z]|^2 = \prod_{i<j}^{N} |z_i - z_j|^{2m} e^{-\frac{1}{2}\sum_{j=1}^{N}|z_j|^2} = e^{-\beta U}$$

$$\beta = \frac{2}{m} \qquad U = m^2 \sum_{i<j} \left(-\ln|z_i - z_j|\right) + \frac{m}{4}\sum_{k} |z_k|^2$$

2D system

$$\int d\mathbf{s} \cdot \mathbf{E} = 2\pi Q \qquad \mathbf{E}(\mathbf{r}) = \frac{Q\hat{r}}{r} \qquad \phi(\mathbf{r}) = Q\left(-\ln\frac{r}{r_0}\right)$$

- Hence, potential energy among a group of objects with charge *m* is

$$U_0 = m^2 \sum_{i<j} \left(\ln|z_i - z_j|\right)$$

- Second term in U (Poissons Equation)

$$-\nabla^2 \frac{1}{4}|z|^2 = -\frac{1}{l_b^2} = 2\pi\rho_B$$

Plasma Analogy (II)

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$$U = m^2 \sum_{i<j} (-\ln|z_i - z_j|) + \frac{m}{4} \sum_k |z_k|^2$$

Potential energy of interaction among a group of objects with charge *m*

Energy of charge *m* objects interacting with negative background

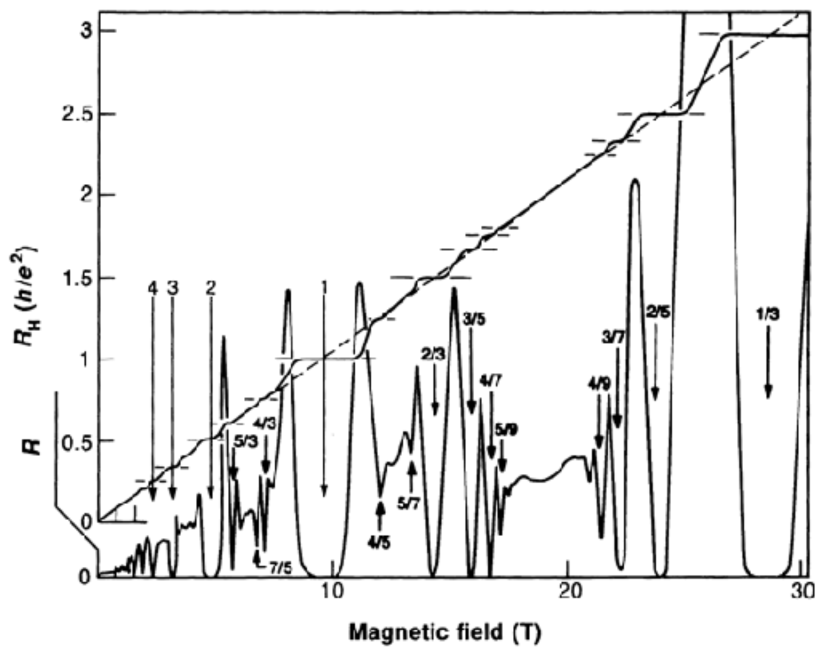
$2\pi l_b^2$ → Area containing one flux quantum → Background charge density B/ϕ_0

Neutrality → $nm + \rho_B = 0$ → $n = \frac{1}{m}\frac{1}{2\pi l_b^2}$

For a filled LL, with *m =1*, this is the correct answer for the density, since every single-particle state is occupied and there is one state per flux quantum

Excitation Gap?

- Every pair of particles has a relative angular momentum greater than or equal to *m*
- Because the relative angular momentum of a pair can change only in discrete (even integer) units it turns out that a hard core, repulsion, model has an excitation gap
- For example for $m = 3$, any excitation out of the Laughlin ground state weakens the nearly ideal correlations by forcing at least one pair of particles to have relative angular momentum *1* instead of *3*.
- This costs energy: hence a gap



Importance

- Two Nobel Prizes
- IQHE 1985 (Klaus von Klitzing);
- FQHE 1998 (Robert Laughlin, Horst Stormer and Daniel Tsui)
- The value of the resistance at the plateaus only depends on fundamental *constants* of physics: electric charge (*e*) and Planck's constant (h)
- It is accurate to 1 part in 100000000
- The IQHE is used as the primary resistance standard (although 1 klitzing (h/e^2) is 25,813 Ohms)

LLL One Body States ($\Omega = 0$)

Harmonic oscillator

$$H_0 = \hbar\omega_\perp \left(a_+^\dagger a_+ + a_-^\dagger a_- + 1\right)$$

$$a_\pm = \frac{a_x \mp i a_y}{\sqrt{2}}$$

$$a_\pm^\dagger = \frac{a_x^\dagger \pm i a_y^\dagger}{\sqrt{2}}$$

$$a_x = \frac{1}{\sqrt{2}}\left(\frac{x}{d_\perp} + i\frac{p_x d_\perp}{\hbar}\right) \quad a_y = \frac{1}{\sqrt{2}}\left(\frac{y}{d_\perp} + i\frac{p_y d_\perp}{\hbar}\right)$$

$$a_x^\dagger = \frac{1}{\sqrt{2}}\left(\frac{x}{d_\perp} - i\frac{p_x d_\perp}{\hbar}\right) \quad a_y^\dagger = \frac{1}{\sqrt{2}}\left(\frac{y}{d_\perp} - i\frac{p_y d_\perp}{\hbar}\right)$$

Angular momentum

$$L_z = x p_y - y p_x = \hbar\left(a_+^\dagger a_+ - a_-^\dagger a_-\right)$$

Create and destroy one quantum with positive (negative) circular polarization and one unit of positive (negative) angular momentum

LLL One Body States ($\Omega \neq 0$)

Rotating system

$$H_0' = H_0 - \Omega L_z = \hbar\omega_\perp + \hbar\left(\omega_\perp - \Omega\right) a_+^\dagger a_+ + \hbar\left(\omega_\perp + \Omega\right) a_-^\dagger a_-$$

Eigenvalues

$$\epsilon\left(n_+, n_-\right) = n_+ \hbar\left(\omega_\perp - \Omega\right) + \hbar n_-\left(\omega_\perp + \Omega\right)$$

Landau Levels

$\Omega \to \omega_\perp$ ⟶ Eigenvalues are essentially independent of n_+

↓

n_- becomes the Landau Level index

LLL One Body States (Ω)

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$$n = n_+ + n_- \qquad\qquad m = n_+ - n_-$$

$$\epsilon\left(\frac{1}{2}(n+m), \frac{1}{2}(n-m)\right) = n\hbar\omega_\perp - m\hbar\Omega$$

- The excitation energy is independent of *m* forming an inverted pyramid of states. For each non-negative integer *n* there are *n +1* degenerate angular momentum states (-*n* … *n,* in steps of 2)
- The degeneracy is lifted
- States become nearly degenerate again, forming essentially horizontal rows.

LLL One Body States ($\Omega \approx 1$)

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LLL Physics appropriate when

$$\Omega/\omega_\perp \approx 1$$

Energy scales

Gap → $2\hbar\omega_\perp$

Interaction energy → $gn(0) = \mu$

→ $\mu/\left(2\hbar\omega_\perp\right) \ll 1$

Eigenfunctions of LLL

$$\psi_m \propto r^m e^{i\phi m} e^{-r^2/(2d_\perp^2)}$$

- *m =0* represents the vacuum for both circularly polarized modes
- The higher states (*m >0*) can be written as

$$\psi_m \propto \zeta^m e^{-r^2/(2d_\perp^2)}, \ \ \text{where } \zeta = (x+iy)/d_\perp$$

Rotation and Magnetic field

$$H_0' = \frac{p^2}{2M} + \frac{1}{2} M\omega_\perp^2 r^2 - \boldsymbol{\Omega} \cdot \mathbf{r} \times \mathbf{p}$$

↓

$$H_0' = \frac{(p - M\Omega \times \mathbf{r})^2}{2M} + \frac{1}{2} M \left(\omega_\perp^2 - \Omega^2\right) r^2$$

Synthetic vector potential

$$q\mathbf{A} \rightarrow M\Omega \times \mathbf{r}$$

$$\Omega = \hat{k}\Omega_z$$

→

$$q\mathbf{A} = M\Omega_z \left(-y\hat{i} + x\hat{j}\right)$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

↓

$$\mathbf{B} = \nabla \times \mathbf{A} = 2\Omega_z M/q$$

Typical Movie (Increasing Ω)

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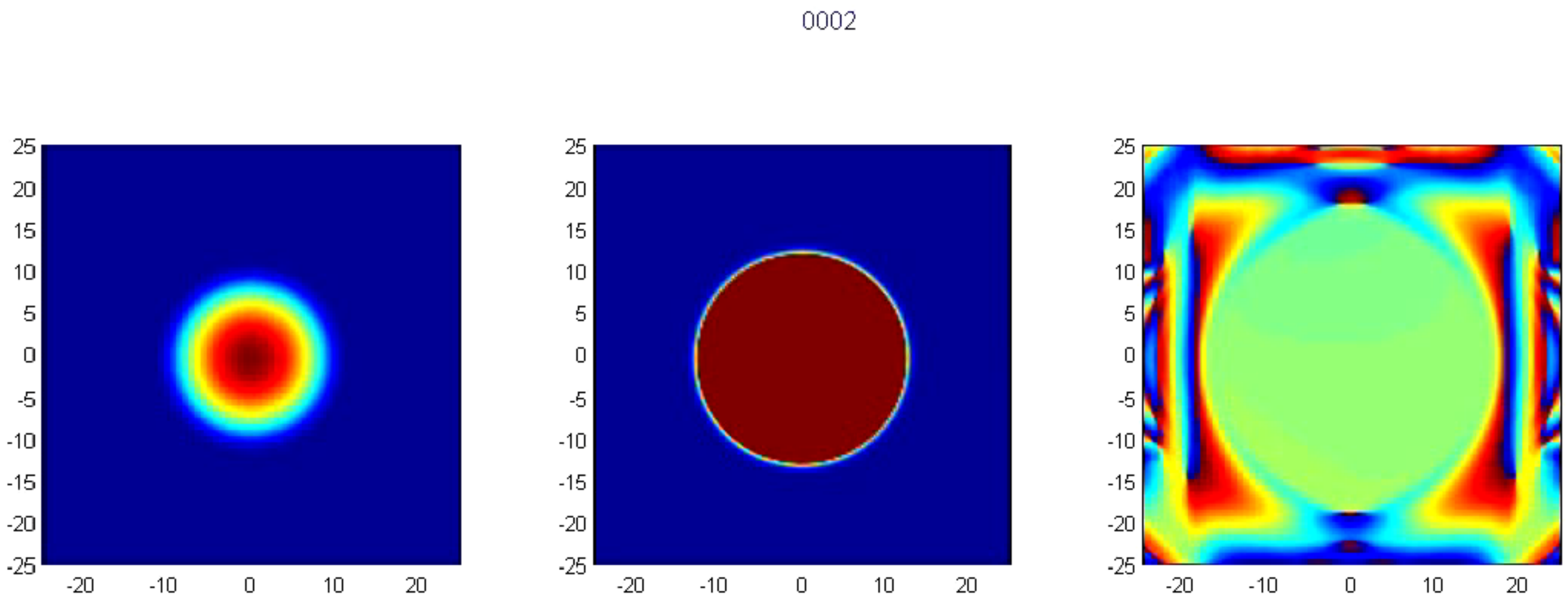
Gross-Pitaevskii Simulation



LLL Condensate Wavefunction

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$$\psi_{LLL} = \sum_{m \geq 0} c_m \psi_m = f(\zeta) e^{-r^2/(2d_\perp^2)}$$

$$f(\zeta) \propto \prod_j (\zeta - \zeta_j)$$

- $f(\zeta)$ vanishes at each of the points ζ_j which are the positions of the nodes of the condensate wave-function
- The phase of this wave-function increases by 2π whenever ζ moves in the positive sense around any of these zeros
- Thus the points ζ_j are precisely the positions of the vortices in the trial state and minimization with respect to the constants c_m is effectively the same as minimization with respect to the position of the vortices: vortex lattice

Energy Minimization

$$E\left[\psi\right] = \int d^2r\psi^* \left(\frac{p^2}{2M} + \frac{1}{2}M\omega_\perp^2 r^2 - \Omega L_z + \frac{1}{2}g_{2D}|\psi|^2\right)\psi$$

$$E\left[\psi_{LLL}\right] = \hbar\Omega + \int d^2r \left[M\omega_\perp^2\left(1 - \frac{\Omega}{\omega_\perp}\right)r^2\left|\psi_{LLL}\right|^2 + \frac{1}{2}g_{2D}|\psi_{LLL}|^4\right]$$

Unrestricted minimization

$$\left|\psi_{min}\right|^2 = n_{min}(0)\left(1 - \frac{r^2}{R_0^2}\right) = \frac{\mu_{min}}{g_{2D}}\left(1 - \frac{r^2 M\omega_\perp^2(1-\tilde{\Omega})}{\mu_{\mathrm{min}}}\right)$$

↓

$$\mu_{\mathrm{min}} = \sqrt{\frac{8aN(1-\tilde{\Omega})}{Z}}, \quad \text{where } Z = 2\pi d_z$$

LLL Condition (Unrestricted)

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$$\mu_{\text{min}} \leq 2\hbar\omega_{\perp}$$

$$1 - \tilde{\Omega} \leq \frac{Z}{2Na}$$

Unrestricted minimization!!

What about vortices?

Highly Correlated States (ν)

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Mean field LLL regime:

$$1 - \tilde{\Omega} \leq \frac{Z}{2N\beta a}, \quad \text{where } \beta = 1.1596$$

- At higher rotation frequencies the meanfield LLL regime should eventually disappear through a quantum phase transition, leading to a different, highly correlated, manybody ground state.
- For meanfield LLL regime

$$N_v \approx \frac{R_0^2}{d_\perp^2} = \sqrt{\frac{8Na\beta}{Z(1-\tilde{\Omega})}}$$

$$\nu = \frac{N}{N_v} = \sqrt{\frac{Z(1-\tilde{\Omega})N}{8a\beta}}$$

Exact Diagonalization ($v \geq v_c$)

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- The equilibrium state in the meanfield LLL regime is a vortex array that breaks the rotational symmetry and is not an eigenstate of L_z
- Could use exact diagonalisation to study the ground state for increasing N_v
- Studies have investigated different filling fractions, v, from 0.5 to 9.
- Comparison between the meanfield LLL energy and exact diagonalization show that the meanfield vortex lattice is a ground state for $v \geq v_c$ ($v_c = 6$)
- Hence the meanfield LLL regime is valid for ($v_c = 1$)

$$1 - \frac{Z}{2N\beta a} \leq \tilde{\Omega} \leq 1 - \frac{8\beta a}{ZN}$$

Exact Diagonalization ($\nu < \nu_c$)

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- The groundstates are rotationally symmetric incompressible vortex liquids that are eigenstates of L_z
- They have close similarities to the bosonic analogs of the Jain sequence of fractional quantum Hall states
- The simplest of these many body ground states is the bosonic Laughlin state

$$\Psi_{Laughlin}\left(\mathbf{r_1}, \mathbf{r_2}, ..., \mathbf{r_N}\right) \propto \prod_{n<n'}^{N} \left(z_n - z_{n'}\right)^2 e^{-\frac{1}{4}\sum_j |z_j|^2}$$

- No off-diagonal long range order and hence no BEC
- The Laughlin state vanishes whenever two particles come together, enforcing the many-body correlations
- The short range two body potential has zero expectation value in this correlated state
- Strong overlap between exact diagonalization and the Laughlin state ($\nu = 1/2$)

Physics of Transition

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- Consider N bosonic particles in a plane, with $2N$ degrees of freedom
- Vortices appear as the system rotates and the corresponding vortex coordinates provide N_v collective degrees of freedom
- For slowly rotating systems the $2N$ particle coordinates provide a convenient description
- In principle, the N_v collective vortex degrees of freedom should reduce the original total $2N$ degrees of freedom to $2N - N_v$, but this is unimportant as long as $N_v << N$
- When N_v becomes comparable with N the depletion of the particle degrees of freedom becomes crucial
- This depletion on the particle degrees of freedom drives the phase transition to a wholly new ground state
- Hence when $v = N/N_v$ is small a transition is expected